

On Thurston's core entropy algorithm

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Abstract

The core entropy of polynomials, recently introduced by W. Thurston, is a dynamical invariant extending topological entropy for real maps to complex polynomials, whence providing a new tool to study the parameter space of polynomials. The base is a combinatorial algorithm allowing for the computation of the core entropy given by Thurston, but without supplying a proof. In this paper, we will describe his algorithm and prove its validity.

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1 Introduction

A central theme of holomorphic dynamics is the understanding of the global dynamics of degree d (≥ 2) complex rational maps and degree d complex polynomials. In the quadratic polynomial case, a rich variety of results were known, see for example [6, 11]. But in the higher degree polynomial case ($d \geq 3$), our overall understanding has remained sketchy and unsatisfying.

Recall that given a continuous map f acting on a compact set X , its *topological entropy* $h(X, f)$ is a quantity that measures the complexity growth of the induced dynamical system. It is essentially defined as the growth rate of the number of itineraries under iteration (see [1]).

The *core entropy* of complex polynomials, to be explained below, was introduced by W. Thurston around 2011 in order to develop a qualitative picture of the connectedness locus of degree d polynomials. The concept of core entropy generalizes the entropy theory of real polynomials, for which monotonicity and continuity go back to Milnor and Thurston [12]. Indeed, the invariant real segment in the real polynomial case is replaced by an invariant tree, which captures all the essential dynamics.

Let $d \geq 2$ be an integer, and f a degree d complex polynomial. The **filled-in Julia set** K_f is the set of points which do not escape to infinity under iteration, and the **Julia set** J_f is the boundary of K_f . A point $c \in \mathbb{C}$ is called a **critical point** of f if $f'(c) = 0$. The **critical set** $\text{crit}(f)$ is defined to be

$$\text{crit}(f) = \{c \in \mathbb{C} \mid f'(c) = 0\},$$

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and the **postcritical set** P_f is defined to be

$$\text{post}(f) = \overline{\{f^n(c) : c \in \text{crit}(f), n \geq 1\}}.$$

In many cases, the filled-in Julia set of f includes a forward invariant, finite topological tree that contains $\text{post}(f)$, called the *Hubbard tree*. In particular, this tree exists if f is **postcritically finite**, i.e., $\#\text{post}(f) < \infty$ (see § 3.1).

Definition 1.1 (core entropy). *The core entropy of f , denoted by $h(f)$, is defined as the topological entropy of the restriction of f to its Hubbard tree, when the tree exists.*

The core entropy yields a tool to study the parameter space of complex polynomials. A fundamental tool in this direction is an effective algorithm allowing for the computation of the core entropy.

Let f be a postcritically finite polynomial of degree $d \geq 2$ and let H_f denote its Hubbard tree. The simplest way to compute the entropy of $f : H_f \rightarrow H_f$ is to compute the *incidence matrix* for the *Markov map* f acting on H_f , and take the logarithm of its leading eigenvalue (see Section 2): however, this requires knowledge of the topology of H_f , and is thus difficult to realize on a computer.

To avoid knowing the topology of the Hubbard tree and the action of f on it, W. Thurston developed a purely combinatorially algorithm (without supplying a proof) using the combinatorial data *critical portraits*. The concept of critical portraits and this entropy algorithm are exhaustively explained in § 3.3 and § 4.3, respectively. Roughly speaking, a postcritically finite polynomial f induces a finite collection of finite subsets of the unit circle

$$\Theta_f = \{\Xi_1, \dots, \Xi_t\}.$$

The algorithm takes Θ_f as input, constructs a non-negative matrix A_f (bypassing f and H_f), and provides as output its Perron-Frobenius leading eigenvalue $\rho(\Theta_f)$.

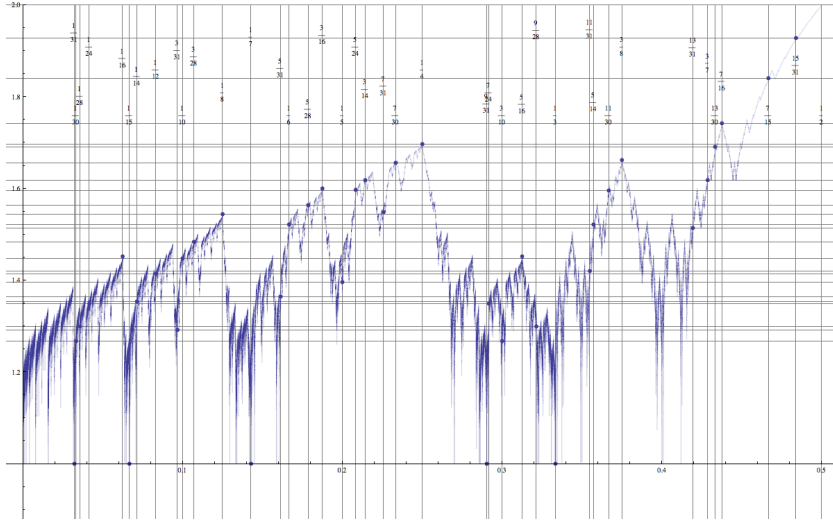


Figure 1: The core entropy of quadratic polynomials, drawn by Thurston

The validity of Thurston's algorithm in the quadratic case was proven by Y. Gao-L. Tan ([20]) and W. Jung ([10]). Based on this algorithm, Tiozzo proved the **continuity conjecture**

of Thurston [18] (Dudko and Schleicher [8] give an alternative proof of the conjecture without using this algorithm): Given $\theta \in \mathbb{Q}/\mathbb{Z}$, the parameter ray of angle θ determines a unique postcritically finite quadratic polynomial $f_{c_\theta} = z^2 + c_\theta$. Let $h(\theta)$ denote the core entropy of f_{c_θ} .

Theorem (Thurston, Dudko-Schleicher, Tiozzo). *The entropy function $h : \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{R}$ extends to a continuous function $\mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$.*

Consequently, by looking at Thurston's entropy graph (Figure 1), one can recover the topological model of the Mandelbrot set (at least for non-renormalizable angles) by taking inscribed horizontal segments.

Analogously, in order to clarify the connectedness locus for degree d ($d \geq 3$) polynomials from the point of view of the core entropy, for example the continuity conjecture in the higher degree case, one should first verify the validity of Thurston's entropy algorithm in the general case. The purpose of this paper is to prove this point. We establish the following main theorem.

Theorem 1.2 (Main). *Let f be a postcritically finite polynomial, and Θ_f a critical portrait of f . Let $\rho(\Theta_f)$ be the output of Thurston's entropy algorithm. Then $\log \rho(\Theta_f)$ equals the core entropy of f , i.e., $\log \rho(\Theta_f) = h(H_f, f)$.*

The organization of the manuscript is as follows. In Section 2 and 3, we introduce some preliminary definitions and results for the topological entropy, Hubbard tree and critical portraits that will be used in the paper. We then provide a detailed description of Thurston's entropy algorithm in Section 4, and prove the main theorem (Theorem 1.2) in Section 5.

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2 Basic results about topological entropy

We will not use the general definition of the topological entropy in the paper (see [1]). Instead, we summarize some basic results about the topological entropy that will be applied below.

We denote by $h(X, f)$ the topological entropy of f on X . The following three propositions can be found in [5].

Proposition 2.1. *If $X = X_1 \cup X_2$, with X_1 and X_2 compact, $f(X_1) \subset X_1$ and $f(X_2) \subset X_2$, then $h(X, f) = \sup(h(X_1, f), h(X_2, f))$.*

Proposition 2.2. *Let Z be a closed subset of X such that $f(Z) \subset Z$. Suppose that for any $x \in X$, the distance of $f^n(x)$ to Z tends to 0, uniformly on any compact set in $X - Z$. Then $h(X, f) = h(Z, f)$.*

Proposition 2.3. *Assume that π is a surjective semi-conjugacy*

$$\begin{array}{ccc} Y & \xrightarrow{q} & Y \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & X. \end{array}$$

Then $h(X, f) \leq h(Y, q)$. Furthermore, if $\sup_{x \in X} \#\pi^{-1}(x) < \infty$ then $h(X, f) = h(Y, q)$.

In this paper, we mainly use the topological entropy for real maps of dimension one.

A (finite topological) **graph** G is a compact Hausdorff space which contains a finite non-empty set V_G (the set of vertices), such that every connected component of $G \setminus V_G$ is homeomorphic to an open interval of the real line. Since any graph can be embedded in \mathbb{R}^3 (see [14]), in what follows we will consider each graph endowed with the topology induced by the topology of \mathbb{R}^3 .

Let X, Y be two topological spaces. A map $\phi : X \rightarrow Y$ is called **monotone** if ϕ is continuous and $\phi^{-1}(y)$ is a connected compact subset of X for every $y \in \phi(X)$.

The following fact will be repeatedly used in the paper. We refer to e.g. [4, A.13] for a proof.

Proposition 2.4. *Let $\phi : [0, 1] \rightarrow X$ be a non-constant monotone map. Then the image $\phi([0, 1])$ is an arc.*

Let G be a finite graph with vertex set V_G . A continuous map $f : G \rightarrow G$ is called **Markov** if there is a finite subset A of G containing V_G such that $f(A) \subset V_G$ and f is monotone on each component of $G \setminus A$. By the definition, an edge of G is mapped either to a vertex of G or the union of several edges of G . Enumerate the edges of G by e_i , $i = 1, \dots, k$. We then obtain an **incidence matrix** $D_{(G,f)} = (a_{ij})_{k \times k}$ of (f, G) such that $a_{ij} = \ell$ if $f(\gamma_i)$ covers e_j precisely ℓ times. Note that choosing different enumerations of the edges gives rise to conjugate incidence matrices, so in particular, the eigenvalues are independent of the choices.

Denote by ρ the greatest non-negative eigenvalue of $D_{(G,f)}$. By the Perron-Frobenius theorem such an eigenvalue exists and equals the growth rate of $\|D_{(G,f)}^n\|$ for any matrix norm.

The following is classical (see [2, 13]):

Proposition 2.5. *The topological entropy $h(G, f)$ is equal to 0 if $D_{(G,f)}$ is nilpotent, i.e., all eigenvalues of $D_{(G,f)}$ are zero; and equal to $\log \rho$ otherwise.*

A special and important type of graph is the (topological) **tree**, which is a connected graph without cycles. A point p of a tree T is called an **endpoint** if $T \setminus \{p\}$ is connected, and called a **branched point** if $T \setminus \{p\}$ has at least 3 connected components. For any two points $p, q \in T$, there is a unique arc in T joining p and q . We denote this arc by $[p, q]_T$.

3 Hubbard trees and critical portraits of postcritically finite polynomials

We here recall some classical results about the Hubbard tree and the critical portraits of a postcritically finite polynomial.

3.1 Postcritically finite polynomials

Let f be a postcritically finite polynomial, i.e., each of its critical points has a finite (and hence periodic or preperiodic) orbit under the action of f . By classical results of Fatou, Julia, Douady and Hubbard, the filled Julia set $K_f = \{z \in \mathbb{C} \mid f^n(z) \not\rightarrow \infty\}$ is compact, connected, locally connected and locally arc-connected. These properties also hold for the Julia set $J_f := \partial K_f$.

The Fatou set $F_f := \overline{\mathbb{C}} \setminus J_f$ consists of one unbounded component $U(\infty)$ which is equal to the basin of attraction of ∞ , together with at most countably many bounded components constituting the interior of K_f . Each of the sets K_f, J_f, F_f and $U(\infty)$ is fully invariant by f ; each Fatou component is (pre)periodic (by Sullivan's non-wandering domain theorem, or by the hyperbolicity of the map); and each periodic Fatou component cycle contains at least one critical point of f (including ∞).

As a consequence, for f a postcritically finite polynomial, there is a system of Riemann mappings

$$\left\{ \phi_U : \mathbb{D} \rightarrow U \mid U \text{ Fatou component} \right\}$$

satisfying that each extends to a continuous map on the closure $\overline{\mathbb{D}}$, so that the following diagram commutes for all U :

$$\begin{array}{ccc} \overline{\mathbb{D}} & \xrightarrow{\text{power map } z^{d_U}} & \overline{\mathbb{D}} \\ \phi_U \downarrow & & \downarrow \phi_{f(U)} \\ \overline{U} & \xrightarrow{f} & \overline{f(U)}. \end{array} \quad (3.1)$$

The image $\phi_U(0)$ is called the **center** of the Fatou component U . It is easy to see that any center is mapped to a critical periodic point by some iterations of f . For a point $z \in F_f$, denote by $U(z)$ the Fatou component containing z . On every periodic Fatou component U , including $U(\infty)$, the map ϕ_U realizes a conjugacy between a power map and the first return map on U . The image in U under ϕ_U of CLOSED radial lines in \mathbb{D} are, by definition, **internal rays** on U if U is bounded and **external rays** if $U = U(\infty)$. Since a power map sends a radial line to a radial line, the polynomial f sends an internal/external ray to an internal/external ray.

If U is a bounded Fatou component, then $\phi_U : \mathbb{D} \rightarrow \overline{U}$ is a homeomorphism and thus every boundary point of U receives exactly one internal ray from U . This is in general not true for $U(\infty)$, where several external rays may land at a common boundary point. For any $\theta \in \mathbb{R}/\mathbb{Z}$, we use $\mathcal{R}_f(\theta)$ or simply $\mathcal{R}(\theta)$ to denote the image by $\phi_{U(\infty)}$ of the radial ray $\{re^{2\pi i\theta}, 0 < r \leq 1\}$ and will call it the external ray of angle θ . We also use $\gamma(\theta) = \phi_{U(\infty)}(e^{2\pi i\theta})$ to denote the **landing point** of the ray $\mathcal{R}(\theta)$.

Definition 3.1 (supporting rays). *We say that an external ray $\mathcal{R}(x)$ supports a bounded Fatou component U if*

1. *the ray lands at a boundary point q of U , and*
2. *there is a sector based at q delimited by $\mathcal{R}(x)$ and the internal ray of U landing at q such that the sector does not contain other external rays landing at q (see Figure 2).*

3.2 The Hubbard trees of postcritically finite polynomials

The material of this part comes from [6, Chpter 2] and [17, Chpter I].

Let f be a postcritically finite polynomial. Then any pair of points in the closure of a bounded Fatou component can be joined in a unique way by a Jordan arc consisting of (at most two) segments of internal rays. We call such arcs **regulated** (following Douady and Hubbard). Since K_f is arc connected, given two points $z_1, z_2 \in K_f$, there is an arc $\gamma : [0, 1] \rightarrow K_f$ such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$. In general, we will not distinguish between the map γ and its

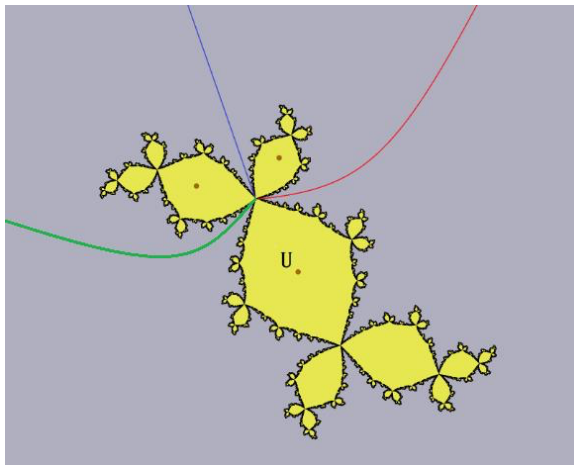


Figure 2: The red and green rays are supporting rays of the Fatou component U , but the blue one is not.

image. It is proved in [6] that such arcs can be chosen in a unique way so that the intersection with the closure of a Fatou component is regulated. We still call such arcs regulated and denote them by $[z_1, z_2]$. We say that a subset $X \subset K_f$ is **allowably connected** if for every $z_1, z_2 \in X$ we have $[z_1, z_2] \subset X$. We define the **regulated hull** of a subset X of K_f to be the minimal closed allowably connected subset of K_f containing X .

Proposition 3.2 ([6]. Proposition 2.7). *For a collection of z_1, \dots, z_n finitely many points in K_f , their regulated hull is a finite tree.*

The **Hubbard tree** of f , denoted by H_f , is defined to be the regulated hull of the finite set $\text{crit}(f) \cup \text{post}(f)$ in K_f . The vertex set V_{H_f} of H_f is the union of $\text{crit}(f) \cup \text{post}(f)$ and the branched points of H_f .

We remark that in most of literatures, the Hubbard tree is defined to be the regulated hull of $\text{post}(f)$, but not $\text{crit}(f) \cup \text{post}(f)$. By Proposition 2.2, these two kinds of definitions of the Hubbard tree give the same core entropy of f . Our definition here is just for convenience of the discussion below. The following is a well-known result (see [17, Chapter I, § 1]).

Proposition 3.3. *Any postcritically finite polynomial f maps each edge of H_f homeomorphically onto the union of several edges of H_f . Consequently, $f : H_f \rightarrow H_f$ is a Markov map.*

Using Proposition 2.5, we relate the topological entropy of f on H_f to the spectral radius of the incidence matrix $D_{(H_f, f)}$ by $h(H_f, f) = \log \rho(D_{(H_f, f)})$.

3.3 Critical portraits of polynomials

For a postcritically finite polynomial, we obtain a collection of combinatorial data from the rays landing at its critical points and on the critical Fatou components, called a *critical portrait*. These combinatorial data were first introduced in [3] to classify the strictly preperiodic polynomials as dynamics, and then extended by Poirier [16] (called critical marking) to the general case (including periodic critical points). Our definition will be less restrictive than that of Poirier.

Let f be a postcritically finite polynomial of degree d . Denote by c_1, \dots, c_m the critical points of f in J_f , and by U_1, \dots, U_n the critical Fatou components of f . We construct a **critical portrait** of f

$$\Theta = \Theta_f := \{\Theta(c_1), \dots, \Theta(c_m); \Theta(U_1), \dots, \Theta(U_n)\}$$

as follows: Let c be a critical point, *resp.* U be a critical Fatou component of f . Denote $\delta_c = \deg_f(c)$, *resp.* $\delta_U = \deg(f|_U)$.

- The case that c is a critical point in J_f . For an angle θ such that $\mathcal{R}(\theta)$ lands at $f(c)$, there are exactly δ_c rays among $f^{-1}(\mathcal{R}(\theta))$ landing at c . We denote $\Theta(c)$ the set of arguments of these rays.
- The case that U is a periodic Fatou component. Let

$$U \mapsto f(U) \mapsto \dots \mapsto f^n(U) = U$$

be a critical Fatou cycle of period n . We will construct the associated set $\Theta(U')$ for every critical Fatou component U' in this cycle simultaneously. Let $z \in \partial U$ be a **root** of U , i.e., a periodic point with period less than or equal to n . Note that this choice naturally determines a root $f^k(z)$ for each Fatou component $f^k(U)$ for $k \in \{0, \dots, n-1\}$, which is called the **preferred root** of $f^k(U)$. Let U' be a critical Fatou component in the cycle and z' its preferred root. We consider a supporting ray $\mathcal{R}(\theta)$ for this component U' at z' . The period of this ray is exactly n . We define $\Theta(U')$ the set of arguments of the $\delta_{U'}$ supporting rays for the component U' that are inverse images of $f(\mathcal{R}(\theta)) = \mathcal{R}(d\theta)$.

- The case that U is a strictly preperiodic Fatou component. Let n be the minimal number such that $f^n(U)$ is a critical Fatou component. And let $z \in \partial U$ be a point which is mapped by f^n to the point $\gamma(\eta)$ with $\eta \in \Theta(f^n(U))$. We consider a supporting ray $\mathcal{R}(\theta)$ for the component U at z , and define $\Theta(U)$ to be the set of arguments of the δ_U supporting rays for the component U that are inverse images of $f^n(\mathcal{R}(\theta))$.

We call $\Theta(c_1), \dots, \Theta(c_m) \in \Theta_f$ the **Julia-type** and $\Theta(U_1), \dots, \Theta(U_n) \in \Theta_f$ the **Fatou-type** elements of Θ_f . In the following, we will use several notations as listed in the table below:

$\Theta_{\mathcal{F}} = \{\Theta(U_1), \dots, \Theta(U_n)\}$	$\text{crit}(\Theta_{\mathcal{F}}) = \cup_{i=1}^n \Theta(U_i)$	$\text{post}(\Theta_{\mathcal{F}}) = \cup_{n \geq 1} \tau^n(\text{crit}(\Theta_{\mathcal{F}}))$
$\Theta_{\mathcal{J}} = \{\Theta(c_1), \dots, \Theta(c_m)\}$	$\text{crit}(\Theta_{\mathcal{J}}) = \cup_{j=1}^m \Theta(c_j)$	$\text{post}(\Theta_{\mathcal{J}}) = \cup_{n \geq 1} \tau^n(\text{crit}(\Theta_{\mathcal{J}}))$
$\Theta = \Theta_{\mathcal{F}} \cup \Theta_{\mathcal{J}}$	$\text{crit}(\Theta) = \text{crit}(\Theta_{\mathcal{F}}) \cup \text{crit}(\Theta_{\mathcal{J}})$	$\text{post}(\Theta) = \text{post}(\Theta_{\mathcal{F}}) \cup \text{post}(\Theta_{\mathcal{J}})$

By the construction, we immediately get the following proposition (see also [16, Chap.I, 3]). Let $\tau : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ be the map defined by $\tau(\theta) = d\theta \pmod{\mathbb{Z}}$.

Proposition 3.4. *Let f be a postcritically finite polynomial of degree d , and*

$$\Theta = \{\Theta(c_1), \dots, \Theta(c_m); \Theta(U_1), \dots, \Theta(U_n)\} \triangleq \{\Xi_1, \dots, \Xi_{m+n}\} \quad (3.2)$$

be a critical portrait of f . Then Θ satisfies the following properties.

1. Each $\tau(\Xi_i), i \in \{1, \dots, m+n\}$, is a singleton.
2. The sets Ξ_1, \dots, Ξ_{m+n} are pairwise **unlinked**, i.e., each Ξ_i is contained in the closure of a component of $\mathbb{R}/\mathbb{Z} \setminus \Xi_j$ for all $j \neq i$.
3. Each $\#\Xi_i \geq 2$, and $\sum_{i=1}^{m+n} \#(\Xi_i - 1) = d - 1$.
4. The sets $\Theta(c_1), \dots, \Theta(c_m)$ are pairwise disjoint.

4 Thurston's entropy algorithm and the core entropy

The purpose of this section is to define the notions in the following diagram and make preparations for proving Theorem 1.2.

$$\begin{array}{ccc}
 \text{critical portrait } \Theta & \xrightarrow{\text{Thurston's entropy algorithm}} & \log \rho(\Theta) \\
 \uparrow & & \parallel \text{ equality by Thm. 1.2} \\
 \text{p.f polynomial } f & \xrightarrow{\text{the core entropy of } f} & h(H_f, f).
 \end{array}$$

4.1 Formal critical portrait

Denote \mathbb{D} the unit disk. By abuse of notation, we will identify a point of $\partial\mathbb{D}$ with its argument in \mathbb{R}/\mathbb{Z} . Then all angles in the circle are considered to be mod 1, i.e. elements of \mathbb{R}/\mathbb{Z} .

The combinatorial information given in Proposition 3.4 for critical portraits of postcritically finite polynomials may be presented in an abstract way, thus giving rise to the concept of the *formal critical portrait*. Recall that the map $\tau : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ is defined by $\tau(\theta) = d\theta \pmod{\mathbb{Z}}$.

Definition 4.1 (formal critical portrait). *A degree d formal critical portrait is a finite collection of finite subsets of the unit circle $\Theta = \{\Xi_1, \dots, \Xi_t\}$ such that:*

1. for each i , the points in Ξ_i are identified by $\theta \mapsto \tau(\theta)$;
2. for any two members Ξ_i, Ξ_j of Θ , their convex hulls $\text{hull}(\Xi_i)$ and $\text{hull}(\Xi_j)$ in the closed unit disc are either disjoint or intersect at one point on $\partial\mathbb{D}$;
3. each $\#\Xi_i \geq 2$, and $\sum_{i=1}^t \#(\Xi_i - 1) = d - 1$.

In other words, a formal critical portrait of degree d is a collection of disjoint leaves and polygons each of whose vertices are identified under $z \mapsto z^d$, with total criticality $d - 1$ (see Figure 3). By Proposition 3.4, a critical portrait induced by a postcritically finite polynomial is a formal critical portrait.

Lemma 4.2. *If $\Theta = \{\Xi_1, \dots, \Xi_t\}$ is a formal critical portrait of degree d , then $\mathbb{D} \setminus (\cup_{k=1}^t \text{hull}(\Xi_k))$ has d complementary components, and each one takes a total arc length $1/d$ on the unit circle.*

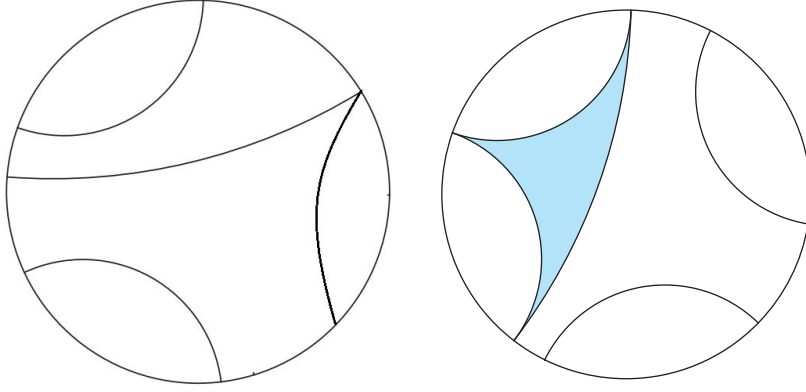


Figure 3: Two formal critical portraits of degree 5

Proof. The set $\text{hull}(\Xi_1)$ chops \mathbb{D} into $\#\Xi_1$ regions. One of them contains $\text{hull}(\Xi_2)$, and is chopped by $\text{hull}(\Xi_2)$ into $\#\Xi_2$ regions. As a result, $\text{hull}(\Xi_1)$ and $\text{hull}(\Xi_2)$ together chop \mathbb{D} into $\#\Xi_1 + (\#\Xi_2 - 1)$ regions. If we chop $\text{hull}(\Xi_3), \dots, \text{hull}(\Xi_s)$ consecutively, at the end we get that the union of convex hulls of Ξ_i 's chop \mathbb{D} into $\#\Xi_1 + \sum_{i=2}^s (\#\Xi_i - 1)$ regions, and the condition 3 says this number is exactly d .

On the other hand, each of these d regions touch the boundary circle in arcs whose total length is a multiple of d , since we know Ξ_i decomposes $\partial\mathbb{D}$ into arcs each of which has length a multiple of $1/d$. Now we have d many union of arcs each of which has total length a multiple of $1/d$, so each one must be of length exactly $1/d$ in order to sum up to 1. \square

Let $\Theta = \{\Xi_1, \dots, \Xi_t\}$ be a formal critical portrait. Recall that the set $\text{crit}(\Theta)$ is defined to be $\text{crit}(\Theta) = \cup_{i=1}^t \Xi_i$.

Definition 4.3 (unlinked equivalent on $\partial\mathbb{D}$). *Two points $x, y \in (\mathbb{R}/\mathbb{Z}) \setminus \text{crit}(\Theta)$ are called unlinked on $\partial\mathbb{D}$ related to Θ if they belong to a common component of $(\mathbb{R}/\mathbb{Z}) \setminus \Xi_i$ for all $\Xi_i \in \Theta$.*

This unlinked relation is an equivalence relation and the following result holds.

Proposition 4.4. *Two points $x, y \in (\mathbb{R}/\mathbb{Z}) \setminus \text{crit}(\Theta)$ are unlinked on $\partial\mathbb{D}$ if and only if they belong to a common complementary component of $\overline{\mathbb{D}} \setminus (\cup_{i=1}^t \text{hull}(\Xi_i))$.*

By this proposition and Lemma 4.2, we see that there are d unlinked equivalence classes on $\partial\mathbb{D}$, denoted by S_1, \dots, S_d , and each one is mapped bijectively by $z \rightarrow z^d$ onto the complement of a finite set in $\partial\mathbb{D}$.

Definition 4.5 (leaf and portrait-leaf). *A leaf is the closure in $\overline{\mathbb{D}}$ of a hyperbolic chord. A leaf is said to be a portrait leaf of a formal critical portrait Θ if it is contained in the boundary of $\text{hull}(\Xi)$ for some $\Xi \in \Theta$. For $x, y \in S^1$, we use \overline{xy} to denote the leaf joining $e^{2\pi ix}$ and $e^{2\pi iy}$.*

Here we remark that the use of hyperbolic chord is nothing special but just for convention, following Thurston. From this view point, a formal critical portrait can be equivalently considered as a collection of portrait leaves and polygons $\{\text{hull}(\Xi_1), \dots, \text{hull}(\Xi_t)\}$.

4.2 Primitive major

We will select a generic subset of degree d formal critical portraits, called the *primitive major* of degree d following Thurston. They will carry the essential combinatorial information of the formal critical portraits.

Definition 4.6 (primitive major). *A degree d formal critical portrait $\Theta = \{\Xi_1, \dots, \Xi_t\}$ is said to be a primitive major if the point 2 in Definition 4.1 is strengthened to be that for any two distinct members Ξ_i, Ξ_j of Θ , their convex hulls in \mathbb{D} are disjoint.*

In Figure 3, the left one is not a primitive major but the right one is. In fact, each formal critical portrait induces a unique primitive major of the same degree. To see this, let $\Theta = \{\Xi_1, \dots, \Xi_t\}$ be a degree d formal critical portrait. We define a relation \diamond on Θ such that $\Xi_i \diamond \Xi_j$ if and only if $\Xi_i \cap \Xi_j \neq \emptyset$. Then the relation \diamond generates an equivalence relation \equiv on Θ , i.e., $\Xi_i \equiv \Xi_j$ if and only if there exist $\Xi_{k_1}, \dots, \Xi_{k_r} \in \Theta$ such that $\Xi_i \diamond \Xi_{k_1} \diamond \dots \diamond \Xi_{k_r} \diamond \Xi_j$, which by inspection is seen to be an equivalence relation. The portrait Θ is therefore divided into the equivalence classes $\mathcal{Q}_1, \dots, \mathcal{Q}_s$. Set

$$\Theta_i := \bigcup_{\Xi \in \mathcal{Q}_i} \Xi \quad \text{for each } i \in \{1, \dots, s\}.$$

The collection of sets $\{\Theta_1, \dots, \Theta_s\}$ is easily checked to be a degree d primitive major, which is called the **primitive major induced by Θ** , and denoted by \mathbf{M}_Θ . From the construction of \mathbf{M}_Θ , we see that a formal critical portrait and its induced primitive major give the same unlinked equivalence relation on $\partial\mathbb{D}$ as described in § 4.1.

In general, we denote by $\mathbf{M} = \{\Theta_1, \dots, \Theta_s\}$ a primitive major of degree d . A portrait leaf of \mathbf{M} is also called a **major leaf**. In order to introduce Thurston's entropy algorithm, we need to consider the position of two points of $\partial\mathbb{D}$ corresponding to a formal critical portrait.

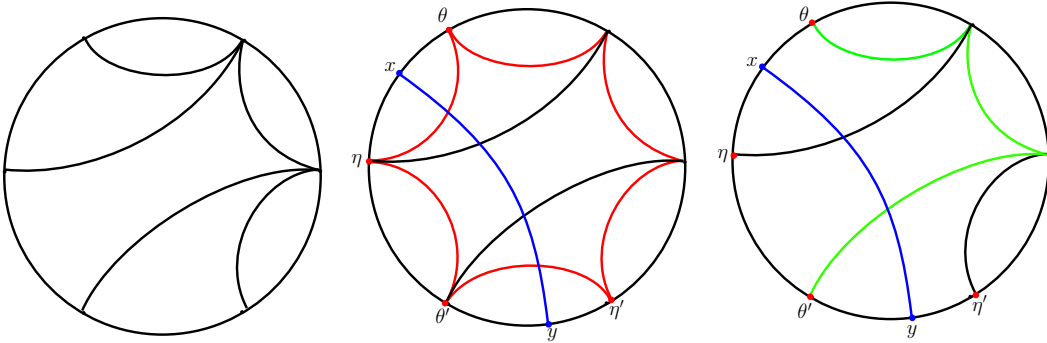


Figure 4: The left figure is a formal critical portrait Θ , and its induced primitive major contains only one element: the polygon with red boundary in the middle figure. The green path in the right figure consists of portrait leaves of Θ and joins θ, θ' .

Let $\Theta = \{\Xi_1, \dots, \Xi_t\}$ be a formal critical portrait. Given two points $x, y \in \mathbb{R}/\mathbb{Z}$ and an element $\Xi \in \Theta$, we say that the points x, y are **separated** by Ξ , if $x, y \notin \Xi$ and the leaf \overline{xy} crosses $\text{hull}(\Xi)$, i.e., $\overline{xy} \cap \text{hull}(\Xi) \neq \emptyset$.

Lemma 4.7. *Let \mathbf{M} be the induced primitive major of Θ . Then two points of $\partial\mathbb{D}$ are separated by an element Θ of \mathbf{M} if and only if they are separated by an element Ξ of Θ contained in Θ .*

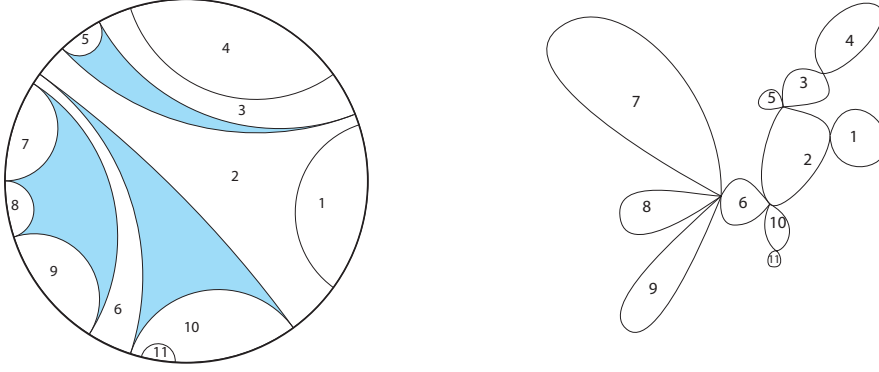


Figure 5: The deformation of the unit circle by shrinking a degree 7 primitive major

Proof. The sufficiency is obvious. For the necessity, assume that $x, y \in \partial\mathbb{D}$ are separated by an element Θ of \mathbf{M} . If $\#\Theta = 2$, then Θ is also an element of Θ , and the conclusion holds. Let $\#\Theta \geq 3$. Then the leaf \overline{xy} intersects two boundary leaves of $\text{hull}(\Theta)$, denoted by $\overline{\theta\eta}$ and $\overline{\theta'\eta'}$. Fixing θ , there is one angle in $\{\theta', \eta'\}$, say θ' , such that any connected subset of \mathbb{D} containing θ and θ' intersects \overline{xy} . Let Ξ and Ξ' be two elements of Θ that contain θ and θ' respectively. By the definition of the equivalence relation \equiv , there exist elements $\Xi_{i_0} := \Xi, \Xi_{i_1}, \dots, \Xi_{i_\ell}, \Xi_{i_{\ell+1}} := \Xi'$ of Θ contained in Θ such that $\Xi_{i_k} \cap \Xi_{i_{k+1}} \neq \emptyset$ for $k \in \{0, \dots, \ell\}$. It follows that the connected set $\cup_{k=0}^{\ell+1} \text{hull}(\Xi_{i_k})$ joins θ and θ' , and then intersects \overline{xy} (see Figure 4). There is thus an element Ξ of Θ among $\{\Xi_{i_0}, \dots, \Xi_{i_{\ell+1}}\}$ fulfilling that $\text{hull}(\Xi) \cap \overline{xy} \neq \emptyset$. Then the conclusion holds. \square

Now let $\mathbf{M} = \{\Theta_1, \dots, \Theta_s\}$ be a primitive major of degree d . Given a pair of ordered points $x, y \in \partial\mathbb{D}$, we say that its **separation set** relative to \mathbf{M} is (k_1, \dots, k_p) if the leaf \overline{xy} successively crosses $\text{hull}(\Theta_{k_1}), \dots, \text{hull}(\Theta_{k_p})$ from x to y with $\Theta_{k_1}, \dots, \Theta_{k_p} \in \mathbf{M}$, and no other elements of \mathbf{M} separate x and y . We say that x and y are **non-separated** by \mathbf{M} if its separation set is empty.

Lemma 4.8. *Two points $x, y \in \partial\mathbb{D}$ are non-separated by \mathbf{M} if and only if there exists an unlinked equivalence class S on $\partial\mathbb{D}$ and two elements Θ_i and Θ_j of \mathbf{M} such that $x, y \in S \cup \Theta_i \cup \Theta_j$. In this case, if x, y do not belong to a common element of \mathbf{M} , then we have $\tau(x) \neq \tau(y)$.*

Proof. Shrink all convex hulls $\text{hull}(\Theta_1), \dots, \text{hull}(\Theta_s)$ to points by a pseudo-isotopy of the plane, then the unit circle deforms to a planar graph constituted by d topological circle, and each of them is the deformation of the closure of an unlinked equivalence class (see Figure 5). Two points of $\partial\mathbb{D}$ are non-separated by \mathbf{M} if and only if their terminal points by this pseudo-isotopy deformation lie in a common topological circle. Thus, two points $x, y \in \partial\mathbb{D}$ are non-separated by \mathbf{M} if and only if there exist an unlinked equivalence class S and two elements Θ_i, Θ_j of \mathbf{M} such that $\overline{S} \cap \Theta_i \neq \emptyset$, $\overline{S} \cap \Theta_j \neq \emptyset$ and $x, y \in \Theta_i \cup S \cup \Theta_j$. We can pick two points $x', y' \in \overline{S}$ such that $\tau(x) = \tau(x')$ and $\tau(y) = \tau(y')$. Whereas within \overline{S} , two points are mapped to a common one by τ if and only if they are connected by a major leaf. We complete the proof. \square

4.3 Thurston's entropy algorithm

In this part, we will show how the algorithm works on a *rational* primitive major.

A formal critical portrait $\Theta = \{\Xi_1, \dots, \Xi_t\}$ is said to be **rational** if all angles in $\text{crit}(\Theta) = \cup_{i=1}^t \Xi_i$ are rational numbers. For instance, a critical portrait induced by a postcritically finite polynomial is rational. If Θ is further a primitive major, we also call it a **rational primitive major**. Throughout this subsection, we fix a rational primitive major

$$\mathbf{M} = \{\Theta_1, \dots, \Theta_s\}$$

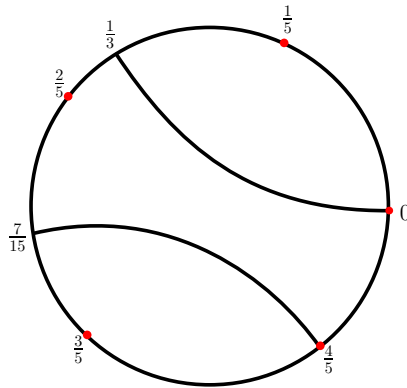
Recall that $\text{crit}(\mathbf{M}) = \cup_{i=1}^s \Theta_i$ and $\text{post}(\mathbf{M}) = \cup_{n \geq 1} \tau^n(\text{crit}(\mathbf{M})) = \cup_{n \geq 1} \cup_{i=1}^s \tau^n(\Theta_i)$. We define a set $O_{\mathbf{M}}$ consisting of all unordered pairs $\{x, y\}$ with $x \neq y \in \text{post}(\mathbf{M})$. The sets $\text{post}(\mathbf{M})$ and $O_{\mathbf{M}}$ are finite. The following is the procedure of Thurston's entropy algorithm acting on the rational primitive major \mathbf{M} .

1. Let $\Sigma_{\mathbf{M}}$ be the abstract linear space over \mathbb{R} generated by the elements of $O_{\mathbf{M}}$.
2. Define a linear map $\mathcal{A}_{\mathbf{M}} : \Sigma_{\mathbf{M}} \rightarrow \Sigma_{\mathbf{M}}$ such that for any basis vector $\{x, y\} \in O_{\mathbf{M}}$,
 - (1) $\mathcal{A}_{\mathbf{M}}(\{x, y\}) = 0$ if x, y belong to a common element Θ of \mathbf{M} ;
 - (2) $\mathcal{A}_{\mathbf{M}}(\{x, y\}) = \{\tau(x), \tau(y)\}$ if x, y are non-separated by \mathbf{M} and do not belong to a common element of \mathbf{M} ; and
 - (3) $\mathcal{A}_{\mathbf{M}}(\{x, y\}) = \{\tau(x), \tau(\Theta_{k_1})\} + \{\tau(\Theta_{k_1}), \tau(\Theta_{k_2})\} + \dots + \{\tau(\Theta_{k_{p-1}}), \tau(\Theta_{k_p})\} + \{\tau(\Theta_{k_p}), \tau(y)\}$ if x, y has the separation set $(k_1, \dots, k_p) \neq \emptyset$.
3. Denote by $A_{\mathbf{M}}$ the matrix of $\mathcal{A}_{\mathbf{M}}$ in the basis $O_{\mathbf{M}}$. It is a non-negative matrix. Compute its leading non-negative eigenvalue $\rho(\mathbf{M})$ (such an eigenvalue exists by the Perron-Frobenius theorem).

Remark 4.9. By Lemma 4.8, we see that $\tau(x) \neq \tau(y)$ in the case 2.(2), and none of the pairs $\{\tau(x), \tau(\Theta_{k_1})\}, \dots, \{\tau(\Theta_{k_p}), \tau(y)\}$ are trivial in case 2.(3).

Definition 4.10. (Thurston's entropy algorithm) Take a rational primitive major \mathbf{M} as input and $\log \rho(\mathbf{M})$ as output. (It is easy to see that $A_{\mathbf{M}}$ is not nilpotent therefore $\rho(\mathbf{M}) \geq 1$).

Example 4.1. Let $\mathbf{M} = \{\{0, 1/3\}, \{7/15, 4/5\}\}$. Then the set $\text{post}(\mathbf{M}) = \{0, 1/5, 2/5, 3/5, 4/5\}$



gives rise to an abstract linear space $\Sigma_{\mathbf{M}}$ with basis:

$$O_{\mathbf{M}} = \left\{ \{0, \frac{1}{5}\}, \{0, \frac{2}{5}\}, \{0, \frac{3}{5}\}, \{0, \frac{4}{5}\}, \{\frac{1}{5}, \frac{2}{5}\}, \{\frac{1}{5}, \frac{3}{5}\}, \{\frac{1}{5}, \frac{4}{5}\}, \{\frac{2}{5}, \frac{3}{5}\}, \{\frac{2}{5}, \frac{4}{5}\}, \{\frac{3}{5}, \frac{4}{5}\} \right\}$$

The linear map $\mathcal{A}_{\mathbf{M}}$ acts on the basis vectors as follows:

$$\begin{aligned} \left\{0, \frac{1}{5}\right\} &\rightarrow \left\{0, \frac{3}{5}\right\}, & \left\{0, \frac{2}{5}\right\} &\rightarrow \left\{0, \frac{1}{5}\right\}, & \left\{0, \frac{3}{5}\right\} &\rightarrow \left\{0, \frac{2}{5}\right\} + \left\{\frac{2}{5}, \frac{4}{5}\right\}, & \left\{0, \frac{4}{5}\right\} &\rightarrow \left\{0, \frac{2}{5}\right\}, \\ \left\{\frac{1}{5}, \frac{2}{5}\right\} &\rightarrow \left\{0, \frac{3}{5}\right\} + \left\{0, \frac{1}{5}\right\}, & \left\{\frac{1}{5}, \frac{3}{5}\right\} &\rightarrow \left\{0, \frac{3}{5}\right\} + \left\{0, \frac{2}{5}\right\} + \left\{\frac{2}{5}, \frac{4}{5}\right\}, & \left\{\frac{1}{5}, \frac{4}{5}\right\} &\rightarrow \left\{0, \frac{3}{5}\right\} + \left\{0, \frac{2}{5}\right\}, \\ \left\{\frac{2}{5}, \frac{3}{5}\right\} &\rightarrow \left\{\frac{1}{5}, \frac{2}{5}\right\} + \left\{\frac{2}{5}, \frac{4}{5}\right\}, & \left\{\frac{2}{5}, \frac{4}{5}\right\} &\rightarrow \left\{\frac{1}{5}, \frac{2}{5}\right\} & \left\{\frac{3}{5}, \frac{4}{5}\right\} &\rightarrow \left\{\frac{4}{5}, \frac{2}{5}\right\}. \end{aligned}$$

We compute $\log \rho(A_{\mathbf{M}}) = 1.395$.

Let Θ be a rational critical portrait. It then induces a unique rational primitive major \mathbf{M}_{Θ} as shown in § 4.2. We consider the entropy algorithm acting on \mathbf{M}_{Θ} as also that on Θ . In this view point, the algorithm works on all rational critical portraits. We simply denote $A_{\Theta} := A_{\mathbf{M}_{\Theta}}$ and $\rho(\Theta) := \rho(\mathbf{M}_{\Theta})$.

4.4 Relating Thurston's entropy algorithm to polynomials

In this part, we will give an intuitive feeling of the relation between the output in the algorithm above and the core entropy of postcritically finite polynomials, and leave the detailed proof to the next section.

Let f be a postcritically finite polynomial of degree $d \geq 2$ and Θ a critical portrait of f as constructed in § 3.3. It is known that the core entropy of f , i.e., the topological entropy of f on H_f , is equal to $\log \rho$ with ρ being the leading eigenvalue of the incidence matrix $D_{(H_f, f)}$ when considering the Markov partition of H_f by its edges. Instead, if one looks at the arcs of H_f between any two postcritical points rather than the edges, the action of f on H_f induces another incidence matrix A_f which turns out to have the same leading eigenvalue ρ .

The advantage of this opinion lies that each postcritical point of f corresponds an angle in $\text{post}(\Theta)$ so that any arc in H_f between postcritical points can be combinatorially represented by an angle pair (not necessary unique). Thus, intuitively, the action of f on these arcs induces a linear map on the space generated by the angle pairs with angles in $\text{post}(\Theta)$, which is the matrix A_{Θ} in the algorithm. Note that the matrix A_{Θ} is in general larger than A_f because one postcritical point of f usually corresponds several angles in $\text{post}(\Theta)$. What we need to do in this paper is to show that these two matrixes have the same leading eigenvalue.

In the quadratic case, a complete proof of Theorem 1.2 can be found in [20, Theorem 13.9]. The idea of the proof is the following:

1. Construct a topological graph G and a Markov action $L : G \rightarrow G$ such that the spectral radius of the incidence matrix $D_{(G, L)}$ is equal to that of A_{θ} (the matrix in the algorithm).
2. Construct a continuous, finite-to-one, and surjective semi-conjugacy Φ from $L : G \rightarrow G$ to $f : H_f \rightarrow H_f$, and then apply Proposition 2.3.

We may thus conclude that

$$\log \rho(A_{\theta}) = \log \rho(D_{(G, L)}) \stackrel{\text{Prop. 2.5}}{=} h(G, L) \stackrel{\text{Prop. 2.3}}{=} h(H_f, f).$$

In the higher degree case, the basic idea is similar, but there are several extra difficulties to overcome. The main problem is that a semi-conjugacy from $L : G \rightarrow G$ to $f : H_f \rightarrow H_f$ is much more difficult to construct, as we will see in the next section. The specific reason will be explained in the rest of this section.

Let f be a postcritically finite polynomial with a critical portrait $\Theta = \{\Xi_1, \dots, \Xi_{m+n}\}$. Let \mathbf{M} be its induced primitive major. Recall that $\text{crit}(\Theta) = \bigcup_{k=1}^{m+n} \Xi_k$ and $\text{post}(\Theta) = \bigcup_{n \geq 1} \tau^n(\text{crit}(\Theta))$. Let G be a topological graph with vertex set

$$V_G := \text{crit}(\Theta) \cup \text{post}(\Theta),$$

and edge set $E_G := \{e(x, y) \mid x \neq y \in V_G\}$ (with $e(x, y) = e(y, x)$). In other words, G is a topological **complete graph** with vertex set V_G .

Mimicking the action of the linear map A_Θ in the algorithm, we define a Markov map $L : G \rightarrow G$ as follows. Let x, y be two distinct vertices in V_G . If x, y belong to a common element of \mathbf{M} , let L map the edge $e(x, y)$ onto the vertex $\tau(x)$. If x, y are non-separated by Θ and do not belong to a common element of \mathbf{M} , let L map $e(x, y)$ monotonously onto the edge $e(\tau(x), \tau(y))$, with x mapped to $\tau(x)$ and y mapped to $\tau(y)$ ($\tau(x) \neq \tau(y)$ by Lemma 4.8). If x, y has the separation set $(k_1, \dots, k_p) \neq \emptyset$, then subdivide the edge $e(x, y)$ into $p+1$ non-trivial arcs $\delta(z_i, z_{i+1}), i \in [0, p]$, with $z_0 := x$ and $z_{p+1} := y$, and let L map each arc $\delta(z_i, z_{i+1})$ monotonously onto the edge $e(\tau(\theta_i), \tau(\theta_{i+1})), i \in [0, p+1]$ ($\tau(\theta_i) \neq \tau(\theta_{i+1})$ by Lemma 4.8), where $\theta_0 := x$, $\theta_{p+1} := y$ and $\theta_i \in \Theta_{k_i}$ for each $i \in [1, p]$.

Consider the subgraph G' of G with vertex set $V_{G'} := \{x \mid x \in \text{post}(\Theta)\}$ and edge set $E_{G'} := \{e(x, y) \in E_G \mid x, y \in V_{G'}\}$. It is easily seen that the edges of G' are in a one-to-one correspondence to the pairs in O_Θ under the correspondence $e(x, y) \rightarrow \{x, y\}$. From the definition of L , we can see that $L(G) \subset G'$ and the incidence matrix of (G', L) is exactly A_Θ . It follows from Propositions 2.2 and 2.5 that

Proposition 4.11. $h(G, L) = h(G', L) = \log \rho(A_\Theta)$.

So we only need to prove $h(G, L) = h(H_f, f)$. Motivated by Proposition 2.3, we try to construct a continuous, finite-to-one, and surjective semi-conjugacy from $L : G \rightarrow G$ to $f : H_f \rightarrow H_f$. *A priori*, the following way seems feasible. Divide the vertices of G into two parts:

$$V_{G, \mathcal{F}} := \text{crit}(\Theta_{\mathcal{F}}) \cup \text{post}(\Theta_{\mathcal{F}}) \quad \text{and} \quad V_{G, \mathcal{J}} := \text{crit}(\Theta_{\mathcal{J}}) \cup \text{post}(\Theta_{\mathcal{J}}), \quad (4.1)$$

where the definitions of $\text{crit}(\Theta_{\mathcal{F}}), \text{post}(\Theta_{\mathcal{F}}), \text{crit}(\Theta_{\mathcal{J}})$ and $\text{post}(\Theta_{\mathcal{J}})$ are given in the table of § 3.3. For simplicity, we call $V_{G, \mathcal{F}}$ the **Fatou-type** vertex set, *resp.* $V_{G, \mathcal{J}}$ the **Julia-type** vertex set. For each $x \in V_{G, \mathcal{F}}$, the ray $\mathcal{R}(x)$ supports a Fatou component, denoted by U_x , whereas for each $y \in V_{G, \mathcal{J}}$, the ray $\mathcal{R}(y)$ lands at a point of $\text{crit}(f) \cup \text{post}(f)$. We are then prompted to first define a map $\chi : V_G \rightarrow \text{crit}(f) \cup \text{post}(f)$ such that $\chi(x)$ is the center of U_x if $x \in V_{G, \mathcal{F}}$ and $\chi(x) = \gamma_f(x)$ if $x \in V_{G, \mathcal{J}}$, and then continuously extend χ to a map from G to H_f , also denoted by χ , such that χ maps each edge $e(x, y)$ monotonously onto the regulated arc $[\chi(x), \chi(y)] \subset H_f$.

This construction of χ indeed works in the quadratic case, so the idea of the proof shown above can be directly realized. However, in the higher degree case, it may happen that a ray $\mathcal{R}(x)$ with certain $x \in V_{G, \mathcal{F}}$ supports simultaneously two Fatou components, or that an angle y belongs to $V_{G, \mathcal{F}} \cap V_{G, \mathcal{J}}$ (see Example 4.2). It means that such χ is not always well defined, so that the construction of the projection from G to H_f described above need not work. This is the key point that causes difficulties of proving Proposition 1.2 in the higher degree case.

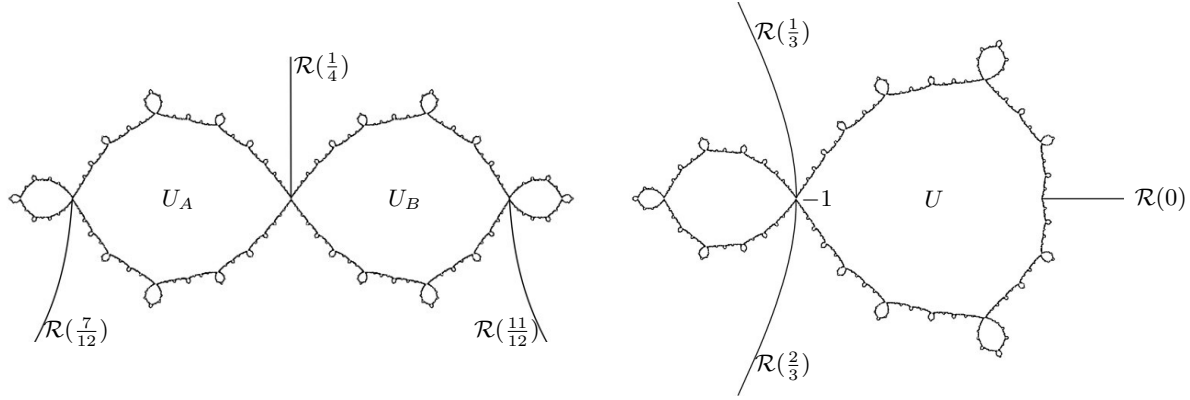


Figure 6: Badly mixed cases of critical portraits

Example 4.2. We first consider the cubic polynomial $f(z) = z^3 - \frac{3}{2}z$. It is easy to see that its critical points are $\pm \frac{\sqrt{2}}{2}$ and they are interchanged by f (see the left one of Figure 6). We may choose a critical portrait of f as

$$\Theta = \left\{ \Theta(U_A) = \{7/12, 1/4\}, \Theta(U_B) = \{11/12, 1/4\} \right\}.$$

The ray $\mathcal{R}(1/4)$ supports the Fatou components U_A and U_B simultaneously.

Consider then the cubic polynomial $f(z) = z^3 + \frac{3}{2}z^2$. One of its critical points 0 is fixed, and another critical point -1 is mapped to a repelling fixed point $1/2$ (the right one of Figure 6). We may choose a critical portrait of f as

$$\Theta = \left\{ \Theta(U) = \{0, 1/3\}; \Theta(-1) = \{1/3, 2/3\} \right\}.$$

In this case, the angle $1/3$ belongs to both $\Theta(U)$ and $\Theta(-1)$.

5 The proof of Theorem 1.2

By Proposition 4.11, we just need to verify the equality $h(G, L) = h(H_f, f)$. The idea of the proof is as follow (using the notations in Section 4).

To solve the problem that $\chi : G \rightarrow H_f$ is not well defined (χ is constructed in § 4.4), we reduce the Hubbard tree H_f to another finite tree T by an equivalence relation such that $\phi \circ \chi : G \rightarrow T$ is well defined, where ϕ denotes the quotient map from H_f to T . Meanwhile, the Markov map $f : H_f \rightarrow H_f$ descends to a Markov map $g : T \rightarrow T$ fulfilling that $\phi \circ f = g \circ \phi$. Then, to establish $h(H_f, f) = h(G, L)$, on one hand we prove that $h(H_f, f) = h(T, g)$, which is based on an analysis of the incidence matrices of (H_f, f) and (T, g) ; on the other hand we verify that $h(T, g) = h(G, L)$, for which we basically use Proposition 2.3. Following this line, we divide our proof into several steps.

Throughout this section, we fix a postcritically finite polynomial f and a critical portrait

$$\Theta = \left\{ \Theta(c_1), \dots, \Theta(c_m); \Theta(U_1), \dots, \Theta(U_n) \right\} \triangleq \{ \Xi_1, \dots, \Xi_{m+n} \}$$

induced by f (see § 3.3).

5.1 Internal rays associated with Θ

Definition 5.1 (admissible internal ray). *Let U be a Fatou component and θ be an angle satisfying that $\mathcal{R}(\theta)$ supports U . Let c_U denote the center of U . We denote by $r_{U,\theta}$ the internal ray of U joining c_U and $\gamma(\theta) \in \partial U$, and call it an admissible internal ray.*

It is clear that the image of an admissible internal ray by f is also an admissible internal ray. More precise, we have $f(r_{U,\theta}) = r_{f(U),\tau(\theta)}$ if $r_{U,\theta}$ is an admissible internal ray.

For each critical Fatou component U and an angle $\theta \in \Theta(U) \in \Theta$, It is known from the construction of Θ that $\mathcal{R}(\theta)$ supports U , and hence $r_{U,\theta}$ is an admissible internal ray. We define a set of admissible internal rays

$$r(\Theta) := \{f^n(r_{U,\theta}) \mid U \text{ is a critical Fatou component, } \theta \in \Theta(U) \in \Theta \text{ and } n \geq 0\}$$

An element of $r(\Theta)$ is called an **internal ray associated with Θ** . By the construction of Θ , we drive some basic properties of such internal rays. Recall that $V_{G,\mathcal{F}} := \text{crit}(\Theta_{\mathcal{F}}) \cup \text{post}(\Theta_{\mathcal{F}})$.

Lemma 5.2. *For any $\theta \in V_{G,\mathcal{F}}$, there exists a critical or postcritical Fatou component U such that $r_{U,\theta}$ is an internal ray associated with Θ . Furthermore, let $r_{U,\theta}$ be an internal ray in $r(\Theta)$. Then we have*

1. $f(r_{U,\theta}) = r_{f(U),\tau(\theta)}$ also belongs to $r(\Theta)$;
2. $r_{U,\theta}$ is periodic if and only if θ is periodic, and they have the same period;
3. suppose that $r_{U,\eta}$ is also an internal ray in $r(\Theta)$, then if U is a critical Fatou component, we get that there exist two angles $\vartheta_1, \vartheta_2 \in \Theta(U)$ such that $r_{U,\theta} = r_{U,\vartheta_1}$ and $r_{U,\eta} = r_{U,\vartheta_2}$; and if $r_{U,\theta}, r_{U,\eta}$ are both periodic, then $r_{U,\theta} = r_{U,\eta}$.

By 1,2 of this lemma, we have that the internal rays associate with Θ are finite and each one is eventually periodic by iterations of f .

5.2 The construction of the quotient map ϕ

By collapsing the internal rays in $r(\Theta)$, we can define a quotient map $\phi : \mathbb{C} \rightarrow \mathbb{C}$, which reduces the Hubbard tree H_f to a finite tree T .

We define a relation \sim on \mathbb{C} such that $z \sim w$ if and only if $z = w$ or z and w are contained in a path constituted by internal rays belonging to $r(\Theta)$. It is clear that \sim is an equivalence relation on \mathbb{C} . We denote by $\phi : \mathbb{C} \rightarrow \mathbb{C}/\sim$ the quotient map.

Lemma 5.3. *The quotient space \mathbb{C}/\sim is homeomorphic to \mathbb{C}*

Proof. Note that each non-trivial \sim -class is a finite tree, the relation \sim clearly satisfies the following 4 properties:

- (1) it is not trivial, meaning that there are at least two distinct equivalence classes;
- (2) it is closed as a subset of $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ equipped with the product topology;

- (3) each equivalence class is a compact connected set;
- (4) the complementary component of each equivalence class is connected.

Moore's Theorem ([15]) says that if an equivalence relation on $\widehat{\mathbb{C}}$ has these 4 properties, then the quotient space of $\widehat{\mathbb{C}}$ modulo this equivalence relation is homeomorphic to $\widehat{\mathbb{C}}$. In our case, we get that \mathbb{C}/\sim is homeomorphic to \mathbb{C} . \square

For simplicity, we assume that $\mathbb{C}/\sim = \mathbb{C}$, and ϕ is chosen to be identity outside a small neighborhood of the filled in Julia set K_f .

Definition 5.4 (essential \sim -equivalence classes). *A \sim -equivalence class ξ is called essential if it is either non-trivial or a point of $\text{crit}(f) \cup \text{post}(f)$ that is disjoint from the non-trivial \sim -equivalence classes.*

Thus $\phi(z) = \phi(w)$ with $z \neq w \in \mathbb{C}$ if and only if z, w belong to an essential \sim -class. Note also that the essential \sim -equivalence classes are pairwise disjoint, and each one is either a finite tree or a point. In the latter case, it is a critical or postcritical point in the Julia set.

We now define $T := \phi(H_f)$. Since each non-trivial \sim -equivalence class ξ are constituted by regulated arcs in K_f , then for any regulated tree H in K_f , we have $[z, w] \subset \xi \cap H$ if $z, w \in \xi \cap H$. The intersection of each ξ and H is hence either empty, or one point or a sub-tree of H . It implies the proposition below.

Proposition 5.5. *The restriction of ϕ on any regulated tree within K_f is monotone. Especially $\phi : H_f \rightarrow T$ is monotone and T is a finite tree in \mathbb{C} .*

We set the vertex set of T to be $V_T = \phi(V_{H_f})$. To characterize the edge set of T , we define a subset $E_{H_f}^{\text{col}}$ of the edge set of H_f by

$$E_{H_f}^{\text{col}} = \{e \in E_{H_f} \mid e \text{ is contained in an essential } \sim\text{-class}\}.$$

Then the edge set of T is equal to $\{\phi(e) \mid e \in E_{H_f} \setminus E_{H_f}^{\text{col}}\}$.

5.3 The essential \sim -equivalence classes

In this part, we prove some results about the essential \sim -equivalence classes, which serve for the next few steps. One may skip this part at the first glance and revisit it when needed.

Lemma 5.6. *For all sufficiently large n and any essential \sim -class ξ , the set $f^n(\xi)$ is either a periodic point in J_f , or a periodic internal ray, or a star-like tree whose unique non-end vertex is a periodic point in J_f and every edge is a periodic internal ray.*

Proof. Let ξ be an essential \sim -class. If ξ is a point, it is a critical or postcritical point in the Julia set, and hence $f^n(\xi)$ is a periodic point in J_f for sufficiently large n .

We now assume that ξ is not a point. It is then the union of several internal rays in $r(\Theta)$. Note that all internal rays in $r(\Theta)$ are eventually periodic by iterations of f , so for any sufficiently large n , the set $f^n(\xi)$ is a tree consisting of periodic internal rays in $r(\Theta)$. By 3 of Lemma 5.2, distinct internal rays in $f^n(\xi)$ lie in different Fatou components. It implies that $f^n(\xi)$ is either a periodic internal ray or a star-like tree with the unique non-end vertex in the Julia set, which must be periodic. \square

We have showed in § 4.2 how a critical portrait induces the unique primitive major of the same degree. From this subsection, we denote by

$$\mathbf{M} := \mathbf{M}_\Theta = \{\Theta_1, \dots, \Theta_s\}$$

the primitive major induced by the critical portrait Θ .

For any element Θ of \mathbf{M} , we define a subset $\text{crit}(\Theta)$ of $\text{crit}(f)$ such that a critical point c belongs to $\text{crit}(\Theta)$ if and only if $\Theta(c) \subset \Theta$ when $c \in J_f$; or $\Theta(U) \subset \Theta$ when c is the center of the Fatou component U , where $\Theta(c)$ and $\Theta(U)$ are elements of the critical portrait Θ (cf. § 3.3 for their construction).

We also define a subset $r(\Theta)$ of the internal rays associated with Θ such that $r_{U,\theta} \in r(\Theta)$ if and only if U is a critical Fatou component with $\Theta(U) \subset \Theta$, and $\theta \in \Theta(U)$. Set

$$H_\Theta := \left(\bigcup_{r_{U,\theta} \in r(\Theta)} r_{U,\theta} \right) \cup \text{crit}(\Theta).$$

Lemma 5.7. *Let Θ be an element of \mathbf{M} , and H_Θ be defined as above. Then*

1. *The set H_Θ is connected, and is hence contained in an essential \sim -equivalence class.*
2. *All external rays with arguments in Θ land at H_Θ and $H_\Theta \cap J_f = \{\gamma(\theta) \mid \theta \in \Theta\}$.*

Proof. If Θ corresponds a trivial \equiv -equivalence class on Θ , i.e., $\Theta = \Theta(c)$ or $\Theta(U) \in \Theta$, the conclusions obviously hold. We then assume in the following that Θ contains at least two elements of Θ . In this case, by 4 of Proposition 3.4, we have $r(\Theta) \neq \emptyset$.

1. For any Fatou critical point $c \in \text{crit}(\Theta)$, it is, by the definition of $\text{crit}(\Theta)$, a center of a critical Fatou component U with $\Theta(U) \subset \Theta$. It follows that an internal ray $r_{U,\theta} \in r(\Theta)$ contains c , where $\theta \in \Theta(U)$. For any Julia critical point $c \in \text{crit}(\Theta)$, we can find a critical Fatou component U such that $\Theta(U) \subset \Theta$ and $\Theta(c) \cap \Theta(U) = \{\theta\} \neq \emptyset$. This is due to point 4 of Proposition 3.4. It follows that the internal ray $r_{U,\theta} \in r(\Theta)$ joins c and the center of U . Therefore, the set H_Θ can be simplified to be $H_\Theta = \bigcup_{r_{U,\theta} \in r(\Theta)} r_{U,\theta}$.

As $r(\Theta) \neq \emptyset$, let $r_{U,\theta}$ and $r_{U',\theta'}$ be two internal rays in $r(\Theta)$, or equivalently, U, U' are two critical Fatou components with $\Theta(U), \Theta(U') \subset \Theta$ and $\theta \in \Theta(U), \theta' \in \Theta(U')$. We need to prove that these two rays can be joined by internal rays in $r(\Theta)$. If $U = U'$, the result is obviously true. Otherwise, there are elements $\Xi_{i_1}, \dots, \Xi_{i_\ell}$ of Θ that are contained in Θ such that

$$\Xi_{i_0} \diamond \Xi_{i_1} \diamond \dots \diamond \Xi_{i_\ell} \diamond \Xi_{i_{\ell+1}},$$

where $\Xi_{i_0} := \Theta(U)$ and $\Xi_{i_{\ell+1}} := \Theta(U')$. For each $k \in \{0, \dots, \ell+1\}$, we denote by c_{i_k} the critical point corresponding to Ξ_{i_k} , i.e., $\Xi_{i_k} = \Theta(c_{i_k})$ or $\Xi_{i_k} = \Theta(U_{i_k})$ with c_{i_k} the center of U_{i_k} . Remember that Ξ_{i_k} is called Julia-type in the former case and Fatou-type in the latter case. When $k \in \{0, \dots, \ell\}$, we set $\{\theta_{i_k}\} := \Xi_{i_k} \cap \Xi_{i_{k+1}}$.

If Ξ_{i_1} is Julia-type, then Ξ_{i_2} is, by 4 of Proposition 3.4, Fatou-type. Consequently, the arc $r_{U_{i_0}, \theta_{i_0}} \cup r_{U_{i_2}, \theta_{i_2}}$ contains the points c_{i_0}, c_{i_1} and c_{i_2} . If Ξ_{i_1} is Fatou-type, it follows that the arc $r_{U_{i_0}, \theta_{i_0}} \cup r_{U_{i_1}, \theta_{i_1}}$ contains the points c_{i_0} and c_{i_1} . Repeating this argument, we find a path constituted by internal rays in $r(\Theta)$ that contains all critical points $c_{i_0}, \dots, c_{i_{\ell+1}}$. This path certainly joins $r_{U,\theta}$ and $r_{U',\theta'}$.

2. It follows directly from the definition of H_Θ and the fact of $r_{U,\theta} \cap J_f = \gamma(\theta)$. \square

5.4 The construction of a Markov map $g : T \rightarrow T$

Clearly, the equivalence relation \sim defined in § 5.2 is f -invariant, i.e., $x \sim y \Rightarrow f(x) \sim f(y)$. The polynomial $f : \mathbb{C} \rightarrow \mathbb{C}$ therefore descends to a map $f/\sim : \mathbb{C} \rightarrow \mathbb{C}$ by ϕ . Set $g := f/\sim$. We have the commutative graph

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f} & \mathbb{C} \\ \phi \downarrow & & \downarrow \phi \\ \mathbb{C} & \xrightarrow{g} & \mathbb{C}. \end{array} \quad (5.1)$$

Note that $g \circ \phi = \phi \circ f$ is continuous, hence g is continuous by the functorial property of the quotient topology.

We call $\text{crit}(g) := \phi(\text{crit}(f))$ the **critical set** of g , and $\text{post}(g) := \phi(\text{post}(f))$ the **postcritical set** of g . For any $\theta \in \mathbb{R}/\mathbb{Z}$, as ϕ is injective within the infinite Fatou component, it maps the external ray $\mathcal{R}_f(\theta)$ to a simple curve, denoted by $\mathcal{R}_g(\theta)$, called the **external ray of g with angle θ** . Similarly, we denote by $\gamma_g(\theta)$ the landing point of $\mathcal{R}_g(\theta)$. Clearly, we have $\gamma_g(\theta) = \phi(\gamma_f(\theta))$ for every $\theta \in \mathbb{R}/\mathbb{Z}$.

With the semi-conjugacy ϕ , the map g inherits many properties of the map f .

- Proposition 5.8.**
1. $\text{post}(g) = \cup_{n \geq 1} g^n(\text{crit}(g))$, $g(V_T) \subset V_T$ and $\text{crit}(g) \cup \text{post}(g) \subset V_T$.
 2. For each $\theta \in \mathbb{R}/\mathbb{Z}$, we have $g(\mathcal{R}_g(\theta)) = \mathcal{R}_g(\tau(\theta))$ and $g(\gamma_g(\theta)) = \gamma_g(\tau(\theta))$.
 3. For each $\theta \in V_G = \text{crit}(\Theta) \cup \text{post}(\Theta)$, the point $\gamma_g(\theta)$ belongs to $\text{crit}(g) \cup \text{post}(g)$.
 4. The tree T is g -invariant and the map $g : T \rightarrow T$ is Markov.

Proof. 1, 2. One just need to remember that $\text{crit}(g)$, $\text{post}(g)$ and V_T are the images of $\text{crit}(f)$, $\text{post}(f)$ and V_{H_f} by ϕ . To complete the argument is straightforward using the commutative graph (5.1).

3. For any $\theta \in V_G$, if $\theta \in V_{\mathcal{F},G}$, by Proposition 5.2, there exists a Fatou component U such that $r_{U,\theta} \in r(\Theta)$ joins $\gamma_f(\theta)$ and a critical or postcritical point of f (the center of U); and if $\theta \in V_{\mathcal{J},G}$, then $\gamma_f(\theta) \in \text{crit}(f) \cup \text{post}(f)$. In both case ϕ maps $\gamma_f(\theta)$ to a point of $\text{crit}(g) \cup \text{post}(g)$.

4. From the commutative graph (5.1), we get $g(T) = \phi \circ f(H_f) \subset \phi(H_f) = T$. By Proposition 3.3, the tree H_f can be broken into a system of arcs Δ_{H_f} such that the restriction of f on each one of Δ_{H_f} is homeomorphic to an edge of H_f . Projected by ϕ , the set of arcs

$$\Delta_T := \{\phi(\delta) \mid \delta \in \Delta_{H_f} \text{ but } \delta \not\subseteq \text{any essential } \sim\text{-class}\}$$

form a decomposition of T . And by the commutative graph (5.1), the map g maps each one of Δ_T either onto one point or monotonously onto an edge of T . Then $g : T \rightarrow T$ is Markov. \square

5.5 The proof of $h(H_f, f) = h(T, g)$

We have shown that $f : H_f \rightarrow H_f$ and $g : T \rightarrow T$ are both Markov maps, so, by Proposition 2.5, it is enough to show that the spectral radii of the incidence matrices $D_{(H_f, f)}$ and $D_{(T, g)}$ are equal. To do this, we need the lemma below.

Recall that $E_{H_f}^{\text{col}} = \{e \in H_f \mid e \text{ is contained in an essential } \sim\text{-class}\}$. By a **cycle** in $E_{H_f}^{\text{col}}$ we mean a subset of $E_{H_f}^{\text{col}}$ of the form

$$O = \{e_0 = e_n, e_1 = f(e_0), \dots, e_n = f(e_{n-1})\}$$

Lemma 5.9. *All edges in $E_{H_f}^{\text{col}}$ are attracted by cycles, i.e., for each $e \in E_{H_f}^{\text{col}}$, the sequence of iterates $f^n(e)$ ($n \geq 0$) eventually falls on the union of cycles in $E_{H_f}^{\text{col}}$.*

Proof. Let $e \in E_{H_f}^{\text{col}}$. It is by definition the union of several internal rays in $r(\Theta)$. So is $f(e)$. It follows that $f(e)$ is the union of several edges in $E_{H_f}^{\text{col}}$.

Since each internal ray $r_{U,\theta} \in r(\Theta)$ contains a point of $\text{crit}(f) \cup \text{post}(f) \subset V_{H_f}$ (the center of U), the edge e is constituted by one or two such rays. Hence $f(e)$ is either still an edge in $E_{H_f}^{\text{col}}$ or the union of two edges in $E_{H_f}^{\text{col}}$ which are both internal rays. Repeating this process, we encounter the following two cases:

- All $f^n(e)$, $n \geq 0$, are edges. In this case e is eventually periodic because H_f is a finite tree.
- For some $k \geq 1$, the set $f^k(e)$ is the union of two edges in $E_{H_f}^{\text{col}}$, which are both internal rays in $r(\Theta)$. Since every such internal ray is eventually periodic, the iteration of e finally falls on the union of two cycles in $E_{H_f}^{\text{col}}$.

This completes the proof of the lemma. □

Proposition 5.10. $h(H_f, f) = h(T, g)$.

Proof. Set $H^{\text{col}} := \cup\{e \mid e \in E_{H_f}^{\text{col}}\}$. We have $f(H^{\text{col}}) \subset H^{\text{col}}$ by Lemma 5.9, and $f : H^{\text{col}} \rightarrow H^{\text{col}}$ is a Markov map. Denoting the incidence matrix of (H^{col}, f) by M , it follows from Proposition 2.5 that $h(H^{\text{col}}, f) = \log \rho(M)$.

Arranging the edges of H_f in the order $E_{H_f}^{\text{col}}, E_{H_f} \setminus E_{H_f}^{\text{col}}$, the incidence matrix of (H_f, f) takes the form

$$D_{(H_f, f)} = \begin{pmatrix} M & \star \\ \mathbf{0} & B \end{pmatrix}$$

where $\mathbf{0}$ denotes a zero-matrix. Note that the matrix B is exactly the incidence matrix of (T, g) , so it is enough to prove $\rho(M) = 1$, or, equivalently, $h(H^{\text{col}}, f) = 0$.

We denote O_1, \dots, O_k the cycles in $E_{H_f}^{\text{col}}$, and set $O = \bigcup_{i=1}^k O_i$. Clearly $f(O) \subset O$ and $f : O \rightarrow O$ is a Markov map. Using Lemma 5.9, Propositions 2.2 and 2.5, we have

$$h(H^{\text{col}}, f) = h(O, f) = \log \rho(D_{(O, f)}).$$

It suffices to show that $\rho(D_{(O, f)}) = 1$.

Group the edges of H_f in O in the order O_1, \dots, O_k . Then the incidence matrix $D_{(O, f)}$ takes the form

$$\begin{pmatrix} M_1 & & \\ & \ddots & \\ & & M_k \end{pmatrix}$$

Fix $j \in [1, k]$. Denote the cycle of edges in O_j by $e_0 \mapsto e_1 \mapsto \dots \mapsto e_n = e_0$. If we further arrange the edges in O_j in the order e_0, e_1, \dots, e_n , the matrix M_j takes the form

$$\begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

Clearly, $\rho(M_j) = 1$. □

5.6 The induced partition of Θ in the f -plane

This and the next subsections will pave the way for proving that $h(G, L) = h(T, g)$. One can also skip this part at the first glance and revisit it when needed.

Recall that

$$\Theta = \{\Theta(c_1), \dots, \Theta(c_m); \Theta(U_1), \dots, \Theta(U_n)\} = \{\Xi_1, \dots, \Xi_{m+n}\}$$

is a critical portrait of the polynomial f , and $\mathbf{M} = \{\Theta_1, \dots, \Theta_s\}$ is the primitive major induced by Θ . As showed in § 4.2, both Θ and \mathbf{M} induce the same unlinked equivalence relation on $\partial\mathbb{D} \setminus \text{crit}(\Theta)$, and the unlinked equivalence classes S_1, \dots, S_d form a partition on $\partial\mathbb{D}$ except for the finite set $\text{crit}(\Theta)$. We introduce in this subsection a corresponding partition on the dynamical plane of f .

Let \overline{xy} be a portrait leaf of Θ , i.e., \overline{xy} is in the boundary of $\text{hull}(\Xi)$ for some element $\Xi \in \Theta$. We define the corresponding *portrait cutting line* as follows. If $x, y \in \Theta(c_i) \in \Theta$, the rays $\mathcal{R}_f(x)$ and $\mathcal{R}_f(y)$ land at the critical point $c_i \in J_f$, and we set

$$\mathcal{R}_f(x, y) := \mathcal{R}_f(x) \cup \mathcal{R}_f(y).$$

If $x, y \in \Theta(U_j) \in \Theta$, the rays $\mathcal{R}_f(x)$ and $\mathcal{R}_f(y)$ support the critical Fatou component U_j , and we set

$$\mathcal{R}_f(x, y) := \mathcal{R}_f(x) \cup r_{U_j, x} \cup r_{U_j, y} \cup \mathcal{R}_f(y).$$

It is clear that each $\mathcal{R}_f(x, y)$ is a simple curve subdividing the complex plane into two open regions. It is called a **portrait cutting line** corresponding to Θ . For each $\Xi_k \in \Theta$, the union of all portrait cutting lines $\mathcal{R}_f(x, y)$ with $x, y \in \Xi_k$ is denoted by $\mathcal{R}_f(\Xi_k)$.

Definition 5.11 (unlinked equivalence in the f -plane). *We say that two points z_1, z_2 of $\mathbb{C} \setminus \bigcup_{k=1}^{m+n} \mathcal{R}_f(\Xi_k)$ are unlinked equivalent in the f -plane if they belong to the same connected component of $\mathbb{C} \setminus \mathcal{R}_f(x, y)$ for all possible portrait cutting lines.*

Looking at the circle at infinity we immediately derive that two rays $\mathcal{R}_f(\theta)$ and $\mathcal{R}_f(\eta)$ without landing points are in a common unlinked equivalence class in the f -plane if and only if θ and η are in a common unlinked equivalence class on $\partial\mathbb{D}$. This provides a canonical correspondence between the unlinked equivalence classes on $\partial\mathbb{D}$ and the one in the f -plane. We then denote V_1, \dots, V_d the unlinked equivalence classes in the f -plane with V_i corresponding to S_i for each $i \in \{1, \dots, d\}$ (see Figure 7). For the statement of the following lemma, we denote by ξ^\cup the union of all essential \sim -equivalence classes.

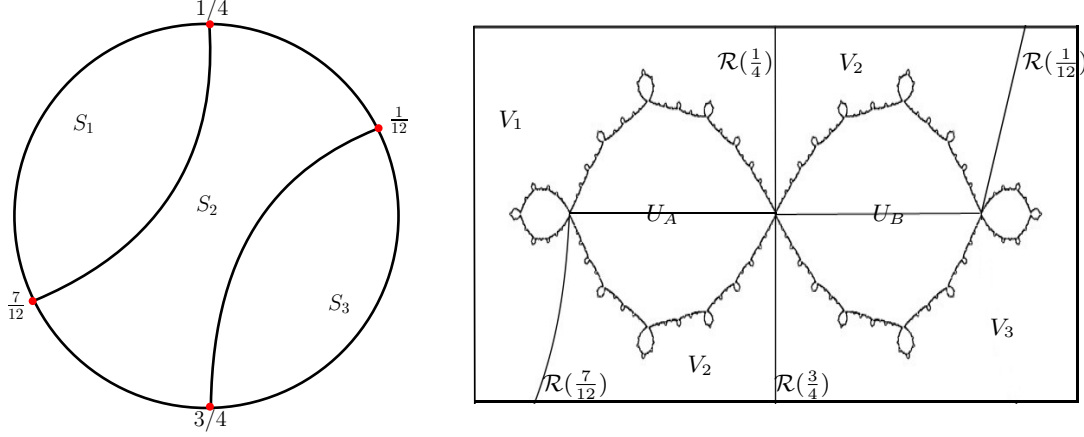


Figure 7: The cubic polynomial $f(z) = z^3 - \frac{3}{2}z$ has a critical portrait $\Theta = \{\Theta(U_A) = \{\frac{1}{4}, \frac{7}{12}\}, \Theta(U_B) = \{\frac{3}{4}, \frac{1}{12}\}\}$. It induces a partition on the unit circle (the left figure), and the corresponding partition in the f -plane (the right picture).

Lemma 5.12. Fix any unlinked equivalence class V in the f -plane and denote its closure by \overline{V} .

1. For two distinct points $z_1, z_2 \in \overline{V} \cap K_f$, the regulated arc $[z_1, z_2] \subset \overline{V}$.
2. The polynomial f maps V bijectively onto a complement of a finite number of external rays and internal rays.
3. If $f(z) = f(w)$ for $z \neq w \in \overline{V} \cap K_f$, then z, w belong to an essential \sim -class.
4. For any $z, w \in \overline{V} \cap K_f$, the restriction of f on $[z, w] \setminus \xi^\cup$ is injective, and its image is contained in a regulated arc in $f([z, w])$.

Proof. 1. We first claim that \overline{V} is connected: otherwise, some portrait cutting line must separate the points of V , which contradicts that V is an unlinked class in the f -plane.

Assume on the contrary that an interior point w of $[z_1, z_2]$ is not in \overline{V} . Then there is a simple curve $R \subset \partial V$ such that any arc joining a point of \overline{V} and w intersects R . Let w_1 be the first, and w_2 be the last points in which the arc $[z_1, z_2]$ intersects R . Since $R \cap K_f$ consists of only internal rays, we have $w \in [w_1, w_2] \subset R \cap K_f \subset \overline{V}$. It contradicts the assumption that $w \notin \overline{V}$.

2. Note that ∂V consists of several external and internal rays, then the image $f(\partial V)$ is the union of a finite number of external and internal rays. Let S be the unlinked equivalence class on $\partial \mathbb{D}$ that corresponds to V . Since τ maps S bijectively onto the complement of $f(\partial S)$ on $\partial \mathbb{D}$, then f maps $V \cap U(\infty)$ bijectively onto the region $U(\infty) \setminus f(\partial V)$. This implies that $f(V) \subset \mathbb{C} \setminus f(\partial V)$.

Fix a point $w_W \in U(\infty)$ in each component W of $\mathbb{C} \setminus f(\partial V)$ and denote by z_W the unique preimage of w_W in $V \cap U(\infty)$. Let w be any point belonging to a connected component W of $\mathbb{C} \setminus f(\partial V)$. Then any arc $\gamma_w \subset W$ joining w and w_W has a unique lift starting from z_W because the lift is contained in V ($\gamma_w \subset W$ so the lift does not intersect ∂V) and V contains no critical points of f . We denote the terminal point of this lift by $h(w)$. Since W is simply-connected, the point $h(w)$ is independent on the choice of γ_w . Hence we get a map $h : \mathbb{C} \setminus f(\partial V) \rightarrow V$. This is easily checked to be the inverse map of $f : V \rightarrow \mathbb{C} \setminus f(\partial V)$.

3. Let z, w be two distinct points in $\bar{V} \cap K_f$ with $f(z) = f(w)$. We first claim that both z and w belong to ∂V . Otherwise we can find two points z' and w' in V near z and w , respectively, such that $f(z') = f(w')$ and this contradicts point 2.

Assume now that $z, w \in \partial V$ and let $[z, w]$ be the regulated arc joining z, w . By point 1 $[z, w] \subset \bar{V}$. Since the intersection of ∂V and K_f consists of only internal rays in $r(\Theta)$, we just need to show that no point of $[z, w]$ are in V . On the contrary, suppose that $x \in V \cap [z, w]$. Note that the image of $[z, w]$ is a tree, so the fact $f(z) = f(w)$ implies that each point in $f([z, w])$ has at least two preimages in $[z, w]$ by f . Then there is a point $y \in [z, w]$ such that $f(x) = f(y)$. It contradicts the claim above (because $x \in V$).

4. The former assertion is a direct consequence of point 3.

Note that the intersection of each \sim -class with $[z, w]$ is either empty, or one point, or a closed segment, so we denote by

$$A_1 := [z_1, w_1], \dots, A_m := [z_m, w_m], \quad m \geq 0,$$

the closures of the connected component of $[z, w] \setminus \xi^\cup$ such that A_i separates A_{i-1} and A_{i+1} for each $i \in [2, m-1]$. Here $m = 0$ represents the case that $[z, w]$ is contained in an essential \sim -class. Obviously, if $m \geq 1$, each A_i is a closed arc. For each $i \in [1, m-1]$, we denote by B_i the connected component of $[z, w] \cap \xi^\cup$ between A_i and A_{i+1} .

For $i = 1, \dots, m$, set $E_i := f(A_i)$. According to item 3, we get that $E_i = [f(z_i), f(w_i)]$ for $i = 1, \dots, m$, and their interior are pairwise disjoint. We also set $F_i := f(B_i)$ for $i = 1, \dots, m-1$. Clearly, each F_i is a tree (maybe reducing to one point) containing $f(w_i)$ and $f(z_{i+1})$.

We claim that except for these two points, the set F_i is disjoint from $\bigcup_{j=1}^m E_j$. To see this, let $p \in B_i$ with $f(p) \in E_j$. Since $E_j = f(A_j)$, there exists a point $q \in A_j$ such that $f(q) = f(p)$. By item 3 and the discussion above, the point q must belong to B_i . It implies that either $j = i$ and $q = w_i$ or $j = i+1$ and $q = z_{i+1}$, which completes the proof of the claim.

Consequently, we denote $P_j := [f(w_j), f(z_{j+1})]$ an arc joining E_j and E_{j+1} for each $j \in \{1, \dots, m-1\}$. Then $P_j \subset F_j$ and $P_j \cap (\bigcup_{i=1}^m E_i) = \{f(w_j), f(z_{j+1})\}$. Therefore $L := (\bigcup_{i=1}^m E_i) \cup (\bigcup_{j=1}^{m-1} P_j)$ is a regulated arc and contained in $f([z, w])$. \square

Remember that $\mathbf{M} = \{\Theta_1, \dots, \Theta_s\}$ is the primitive major induced by Θ . For each $\Theta_i \in \mathbf{M}$, we denote $\mathcal{R}_f(\Theta_i)$ the union of all $\mathcal{R}_f(\Xi)$ with $\Xi \in \Theta$ and $\Xi \subset \Theta_i$.

Lemma 5.13. *Let Θ be an element of \mathbf{M} , then we have*

1. *the intersection of $\mathcal{R}_f(\Theta)$ and K_f is H_Θ (defined in § 5.3);*
2. *if $x, y \in V_G$ are separated by Θ , then the regulated arc $[\gamma_f(x), \gamma_f(y)]$ intersects H_Θ .*

Proof. 1. According to the definition of portrait cutting lines, the intersection of $\mathcal{R}_f(\Theta)$ and K_f is $(\bigcup_{r_{U,\theta} \in r(\Theta)} r_{U,\theta}) \cup \{c \mid \Theta(c) \in \Theta \text{ is contained in } \Theta\}$. It is by definition H_Θ .

2. Without loss of generality, we assume that $\gamma_f(x), \gamma_f(y) \notin \{\gamma_f(\theta) \mid \theta \in \Theta\}$ (otherwise the conclusion holds by 2 of Lemma 5.7). Looking at the circle at infinity, we have that the rays $\mathcal{R}_f(x)$ and $\mathcal{R}_f(y)$ without landing points lie in different components of $\mathbb{C} \setminus \mathcal{R}_f(\Theta)$. It follows that $[\gamma_f(x), \gamma_f(y)] \cap \mathcal{R}_f(\Theta) = [\gamma_f(x), \gamma_f(y)] \cap H_\Theta$ is non-empty. \square

5.7 The induced partition of Θ in the g -plane

Using the projection ϕ , we can descend the quantities defined above in the dynamical plane of f to the dynamical plane of g .

For a portrait leaf \overline{xy} of Θ , we define the corresponding portrait cutting line in the g -plane by $\mathcal{R}_g(x, y) := \phi(\mathcal{R}_f(x, y))$. It is easily checked that $\mathcal{R}_g(x, y) = \mathcal{R}_g(x) \cup \mathcal{R}_g(y)$.

Remember that $\mathbf{M} = \{\Theta_1, \dots, \Theta_s\}$ is the primitive major induced by Θ . Let $\Theta \in \mathbf{M}$. By Lemma 5.7, all external rays in the g -plane with arguments in Θ land at a common critical point $c_\Theta := \phi(\text{crit}(\Theta)) = \phi(H_\Theta)$ of g . Following 1 of Lemma 5.13, we then have

$$\mathcal{R}_g(\Theta) := \phi(\mathcal{R}_f(\Theta)) = \cup_{\theta \in \Theta} \mathcal{R}_g(\theta) \quad (5.2)$$

Similarly, we can also define the unlinked equivalence relation in the g -plane such that two points $z, w \in \mathbb{C} \setminus \cup_{i=1}^s \mathcal{R}_g(\Theta_i)$ are said to be **unlinked equivalence** in the g -plane if they belong to a common component of $\mathbb{C} \setminus \mathcal{R}_g(x, y)$ for all possible portrait cutting lines $\mathcal{R}_g(x, y)$. The following result is straightforward.

Proposition 5.14. *Set $W_i := \phi(V_i), i = 1, \dots, d$. Then each W_i is an unlinked equivalence class in the g -plane. And we have $\overline{W_i} = \phi(\overline{V_i})$ and $\partial W_i = \phi(\partial V_i)$.*

With these preparations, we can prove the key lemma below. Note that for any angle $x \in V_G$, by 3 of Proposition 5.8, the landing point $\gamma_g(x)$ of the ray $\mathcal{R}_g(x)$ belongs to T .

Lemma 5.15. *Let x, y be two distinct points in V_G .*

1. *If x, y are separated by $\Theta \in \mathbf{M}$, then $[\gamma_g(x), \gamma_g(y)]_T$ intersects the critical point $c_\Theta := \phi(\text{crit}(\Theta)) = \phi(H_\Theta)$ of g .*
2. *If x, y are non-separated by Θ , then $\gamma_g(x), \gamma_g(y)$ belong to the closure of an unlinked equivalence class in the g -plane, and g maps $[\gamma_g(x), \gamma_g(y)]_T$ monotonously onto $[\gamma_g(\tau(x)), \gamma_g(\tau(y))]_T$ (We allow $[\gamma_g(x), \gamma_g(y)]_T$ or $[\gamma_g(\tau(x)), \gamma_g(\tau(y))]_T$ to be reduced to one point).*

Proof. 1. Note that ϕ is monotonous when restricted on $[\gamma_f(x), \gamma_f(y)]$ (Proposition 5.5), so $\phi([\gamma_f(x), \gamma_f(y)]) = [\gamma_g(x), \gamma_g(y)]_T$ by Proposition 2.4. Following Lemmas 5.13 and 5.7, the regulated arc $[\gamma_f(x), \gamma_f(y)]$ intersects H_Θ and $\phi(H_\Theta)$ is a singleton, so the conclusion holds.

2. Assume on the contrary that no closures of unlinked equivalence classes in the g -plane contain $\gamma_g(x)$ and $\gamma_g(y)$ simultaneously. By the definition of unlinked equivalence in the g -plane, there exists a portrait cutting line $\mathcal{R}_g(\theta, \eta)$ such that $\gamma_g(x)$ and $\gamma_g(y)$ belong to different components of $\mathbb{C} \setminus \mathcal{R}_g(\theta, \eta)$. Let $\theta, \eta \in \Theta \in \mathbf{M}$. Since all external rays of g with arguments in Θ land together, then the angles x, y do not belong to Θ . It follows that x, y are separated by Θ , which contradicts the condition of the lemma.

To prove the latter result of 2, note that $g(\gamma_g(\theta)) = \gamma_g(\tau(\theta))$ for any $\theta \in \mathbb{R}/\mathbb{Z}$ (Proposition 5.8), so, using Proposition 2.4, we just need to prove that $g|_{[\gamma_g(x), \gamma_g(y)]}$ is monotone.

Let W be an unlinked equivalence class in the g -plane with $\gamma_g(x), \gamma_g(y) \in \overline{W}$. Since $\gamma_g(x), \gamma_g(y)$ also belong to T , there exist two points $z, w \in \overline{V} \cap H_f$ such that $\phi(z) = \gamma_g(x)$ and $\phi(w) = \gamma_g(y)$, where V is the unlinked equivalence class in the f -plane with $\phi(V) = W$. As $\phi|_{[z, w]}$ is monotone (Proposition 5.5), then, by Proposition 2.4, we have $\phi([z, w]) = [\gamma_g(x), \gamma_g(y)]_T$.

We apply the formula $\phi \circ f = g \circ \phi$ on $[z, w]$. We will use the notations in the proof of 3 of Lemma 5.12. Note first that we have $[\gamma_g(x), \gamma_g(y)]_T = \phi([z, w]) = \phi(\bigcup_{i=1}^m A_i)$ and

$$g([\gamma_g(x), \gamma_g(y)]_T) = \phi \circ f(\bigcup_{i=1}^m A_i) = \phi(\bigcup_{i=1}^m E_i). \quad (5.3)$$

Let y be any point of $g([\gamma_g(x), \gamma_g(y)]_T)$. We will show that $(g|_{[\gamma_g(x), \gamma_g(y)]_T})^{-1}(y)$ is connected. Since $\phi|_{H_f}$ is monotone, the fiber $Y := (\phi|_{H_f})^{-1}(y)$ is connected. Denote by L the regulated arc containing all $E_i, i = 1, \dots, m$ as in 3 of Lemma 5.12. Then

$$Y \cap (\bigcup_{i=1}^m E_i) = (Y \cap L) \cap (\bigcup_{i=1}^m E_i) = R \cap (\bigcup_{i=1}^m E_i), \quad (5.4)$$

with $R := Y \cap L$. It is apparent that R is a segment in L . There is thus a segment R' in $[z, w]$ such that $f(R' \cap (\bigcup_{i=1}^m A_i)) = R \cap (\bigcup_{i=1}^m E_i)$. Since f is injective on $\bigcup_{i=1}^m A_i$, then

$$f^{-1}(R \cap (\bigcup_{i=1}^m E_i)) \cap (\bigcup_{i=1}^m A_i) = R' \cap (\bigcup_{i=1}^m A_i). \quad (5.5)$$

By (5.3), (5.4) and (5.5), the set $\phi(R' \cap \bigcup_{i=1}^m A_i)$ is exactly the fiber $(g|_{[\gamma_g(x), \gamma_g(y)]_T})^{-1}(y) = g^{-1}(y) \cap [\gamma_g(x), \gamma_g(y)]_T$. And note that it is either a point or a closed segment. Thus, $g|_{[\gamma_g(x), \gamma_g(y)]_T}$ is monotone as we wishes to show. \square

5.8 The construction of a projection $\Phi : G \rightarrow T$

Recall that G is a topological completed graph with the vertex set $V_G = \text{crit}(\Theta) \cup \text{post}(\Theta)$. By point 3 of Proposition 5.8, each external ray in the g -plane with argument in V_G lands at a point of $\text{crit}(g) \cup \text{post}(g)$. So we define a map $\Phi : V_G \rightarrow \text{crit}(g) \cup \text{post}(g)$ with $\Phi(x) = \gamma_g(x)$ for $x \in V_G$, where $\gamma_g(x)$ denotes the landing point of $\mathcal{R}_g(x)$.

Lemma 5.16. *The map Φ is surjective, and satisfies $g \circ \Phi = \Phi \circ \tau$.*

Proof. By the construction of Θ , for each point $z \in \text{crit}(f) \cup \text{post}(f)$, there exists an angle $x \in V_G$ such that either $\gamma_f(x) = z$ or an internal ray $r_{U,x} \in r(\Theta)$ joins z and $\gamma_f(x)$. In both cases, we have $\phi(z) = \phi(\gamma_f(x)) = \gamma_g(x) = \Phi(x)$. Notice that $\text{crit}(g) \cup \text{crit}(f) = \phi(\text{post}(g) \cup \text{post}(f))$, then Φ is surjective. By point 2 of Proposition 5.8, we get $g \circ \Phi(x) = g(\gamma_g(x)) = \gamma_g(\tau(x)) = \Phi \circ \tau(x)$ for each $x \in V_G$. \square

We continuously extend Φ to a map, also denoted by Φ , from G to T such that Φ maps the edge $e(x, y)$ monotonously onto $[\Phi(x), \Phi(y)]_T$. The case that $\Phi(x) = \Phi(y)$ may happen, in which $[\Phi(x), \Phi(y)]_T$ reduces to a point and Φ maps $e(x, y)$ to the point. We thus obtain a projection from G to T .

Proposition 5.17. *The projection $\Phi : G \rightarrow T$ is surjective.*

Proof. Remember that V_{H_f} consists of the branched points of H_f , $\text{crit}(f)$ and $\text{post}(f)$, so the endpoints of H_f belong to $\text{crit}(f) \cup \text{post}(f)$. Since ϕ maps the endpoints of H_f onto the ones of T , the endpoints of T belong to $\text{crit}(g) \cup \text{post}(g)$. Note that G is a complete graph, so $[p, q]_T \subset \Phi(G)$ for any endpoints p, q of T . We only need to invoke the fact that each edge of T is contained in a regulated arc $[p, q]$ with p, q two endpoints. \square

In the construction, the projection Φ is only required to be monotonously on each edge of G , so it is not necessarily a semi-conjugacy from $L : G \rightarrow G$ to $g : T \rightarrow T$. One may ask whether we can impose some additional conditions on the extension such that Φ becomes a semi-conjugacy? The answer might be yes, but we will not work on this. One reason is that to change Φ into a semi-conjugacy, we need to carefully analyse where g is not injective on each edge of T and correspondingly modifying L piecewise on each edge of G , which is a tedious work. The other reason, which is crucial, is that even if we modify Φ to a semi-conjugacy, it is generally not finite to one, because we can only require that Φ is monotone, but not homeomorphic, restricted on each edge of G , so that Proposition 2.2 is not available.

Instead, we will suitably modify Φ on each edge of G such that the relation $g \circ \Phi = \Phi \circ f$ is satisfied in a weaker sense (see Lemma 5.19 below).

Recall the action of L defined in § 4.4. If we set $e(x, x) := x$ for each $x \in V_G$ and set that a separation set with the form (k_1, \dots, k_0) is empty, then the action of L on an edge $e(x, y) \in E_G$ can be uniformly expressed as follows:

Let the separation set of an ordered pair x, y be $(k_1, \dots, k_p), p \geq 0$, i.e., the leaf \overline{xy} successively crosses $\text{hull}(\Theta_{k_1}), \dots, \text{hull}(\Theta_{k_p})$ from x to y , and no other elements of \mathbf{M} separate x and y . Subdivide the edge $e(x, y)$ into $p + 1$ non-trivial arcs $\delta(z_i, z_{i+1}), i \in [0, p]$, with $z_0 := x$ and $z_{p+1} := y$. And then let L map each arc $\delta(z_i, z_{i+1})$ monotonously onto $e(\tau(\theta_i), \tau(\theta_{i+1}))$, where $\theta_0 := x, \theta_{p+1} := y$ and $\theta_i \in \Theta_{k_i}$ for each $i \in [1, p]$.

Definition 5.18 (subdivision arcs of G). *For any edge $e(x, y)$ of G , a non-trivial arc $\delta(z_i, z_{i+1}) \subset e(x, y)$ described above is called a subdivision arc of G .*

We denote by Δ_G the set of subdivision arcs of G . For a subdivision arc of G , we see from the action of L that it is mapped either monotonously onto an edge of G or onto a vertex of G .

Lemma 5.19. *We can modify the projection Φ on each subdivision arc δ of G such that the following equation holds:*

$$g \circ \Phi(\delta) = \Phi \circ L(\delta).$$

Proof. Let $e(x, y)$ be any edge of G , and (x, y) have the separation set (k_1, \dots, k_p) with $p \geq 0$. Then $e(x, y)$ contains $p + 1$ subdivision arcs $\delta(z_i, z_{i+1}), i = 0, \dots, p$, with $z_0 := x$ and $z_{p+1} := y$. We set $\theta_0 := x, \theta_{p+1} := y$, and pick an angle θ_i in each $\Theta_{k_i} \in \mathbf{M}$.

By 1 of Lemma 5.15, the arc $[\Phi(x), \Phi(y)]_T = [\gamma_g(x), \gamma_g(y)]_T$ (possibly reduced to a point) successively passes through the points

$$c_{\Theta_{k_i}} = \phi(H_{\Theta_{k_i}}) = \gamma_g(\theta_i) = \Phi(\theta_i), \quad i = 1, \dots, p.$$

It follows that $[\Phi(x), \Phi(y)]_T$ also contains $p + 1$ successive subdivision sets $[\Phi(\theta_i), \Phi(\theta_{i+1})]_T, i = 0, \dots, p$, where each of them is either an arc or a point.

We now modify Φ on each subdivision arc $\delta(z_i, z_{i+1})$ of G such that Φ maps $\delta(z_i, z_{i+1})$ to the point $\Phi(\theta_i)$ if $\Phi(\theta_i) = \Phi(\theta_{i+1})$, and maps $\delta(z_i, z_{i+1})$ homeomorphically onto $[\Phi(\theta_i), \Phi(\theta_{i+1})]_T$, otherwise. Note that each pair $\theta_i, \theta_{i+1} \in V_G$ are non-separated by \mathbf{M} , then it follows from 2 of Lemma 5.15 that

$$g[\Phi(\theta_i), \Phi(\theta_{i+1})]_T = g[\gamma_g(\theta_i), \gamma_g(\theta_{i+1})]_T = [\gamma_g(\tau(\theta_i)), \gamma_g(\tau(\theta_{i+1}))]_T = [\Phi(\tau(\theta_i)), \Phi(\tau(\theta_{i+1}))]_T$$

Therefore, after this modification, we have

$$\begin{aligned} g \circ \Phi(\delta(z_i, z_{i+1})) &= g[\Phi(\theta_i), \Phi(\theta_{i+1})]_T = [\Phi(\tau(\theta_i)), \Phi(\tau(\theta_{i+1}))]_T \\ &= \Phi(e(\tau(\theta_i), \tau(\theta_{i+1}))) = \Phi \circ L(\delta(z_i, z_{i+1})) \end{aligned}$$

□

5.9 The construction of a projection $\Psi : \Gamma \rightarrow T$

The aim of this subsection is to construct a quotient graph Γ of G and a finite to one map $\Psi : \Gamma \rightarrow T$.

After the modification of Φ following Lemma 5.19, the restriction of Φ on each subdivision arc is either injective or a constant map.

Definition 5.20 (collapsing subdivision arc). *A subdivision arc of G is called a collapsing subdivision arc if Φ is a constant map on this arc.*

Intuitively, by collapsing each collapsing subdivision arc to one point, we obtain a quotient graph Γ , and the projection $\Phi : G \rightarrow T$ descends to a projection $\Psi : \Gamma \rightarrow T$, which is injective on each edge of Γ . We explain these facts in the following.

Recall that Δ_G denotes the set of all subdivision arcs of G . We denote by Δ_G^{col} the set of all collapsing subdivision arcs of G . We define a relation \simeq on G such that $p \simeq q$ if and only if either $p = q$ or p and q are contained in a path constituted by collapsing subdivision arcs of G . This relation is obviously an equivalence relation, so we denote by $\Gamma := G/\simeq$ the quotient space and by $\wp : G \rightarrow \Gamma$ the quotient map.

Proposition 5.21. *The topological space Γ is a topological graph, and the Markov map $L : G \rightarrow G$ descends to a Markov map $Q : \Gamma \rightarrow \Gamma$ by \wp .*

$$\begin{array}{ccc} G & \xrightarrow{L} & G \\ \wp \downarrow & & \downarrow \wp \\ \Gamma & \xrightarrow{Q} & \Gamma. \end{array} \quad (5.6)$$

Proof. We define $V_\Gamma := \wp(V_G)$ the vertex set of Γ and $E_\Gamma := \{\wp(e) \mid e \in E_G \setminus E_G^{\text{col}}\}$ the edge set of Γ , where E_G^{col} denotes the set of the edges of G which are constituted by collapsing subdivision arcs. It is not difficult to check that the topological space Γ with vertex set V_Γ and edge set E_Γ satisfies the properties of being a topological graph (see Section 2).

Let δ be any collapsing subdivision arc of G . By Lemma 5.19, we have that $\Phi(L(\delta)) = g(\Phi(\delta))$ is a singleton. Then $L(\delta)$ is either a point or the union of some collapsing subdivision arcs. It means that the equivalence relation \simeq is L -invariant. The map $L : G \rightarrow G$ hence descends to a continuous self map on Γ , which is denoted by Q .

Clearly, the arcs $\wp(\delta), \delta \in \Delta_G \setminus \Delta_G^{\text{col}}$, forms a system of subdivision arcs of Γ . By the formula $\wp \circ L = Q \circ \wp$ on G , the restriction of Q on such a subdivision arc is either monotonously onto an edge of Γ , or a constant map. Hence $Q : \Gamma \rightarrow \Gamma$ is a Markov map. □

Proposition 5.22. *There exists a surjective and finite to one map $\Psi : \Gamma \rightarrow T$ such that $\Psi \circ \wp = \Phi$ pointwise on G .*

$$\begin{array}{ccc} G & & \\ \Phi \downarrow & \searrow \wp & \\ & \Gamma & \\ & \swarrow \Psi & \\ & T & \end{array} \quad (5.7)$$

Proof. For any $p \in \Gamma$, we define $\Psi(p) = \Phi \circ \wp^{-1}(p)$. Note that $\wp^{-1}(p)$ is either a point or a connected set consisting of collapsing subdivision arcs. Since Φ maps each collapsing subdivision arc to a point, $\Phi(\wp^{-1}(p))$ is always a singleton. It means that Ψ is well defined.

Let p be a point of T . Since $\Phi : G \rightarrow T$ is surjective (Proposition 5.17), then there is a point $a \in G$ with $\Phi(a) = p$. By the commutative graph (5.7), the point $\wp(a)$ is a preimage of p by Ψ . Hence Ψ is surjective.

To prove that Ψ is finite to one, it is enough to show that its restriction on each edge of Γ is injective. Let $e \in E_\Gamma$, $p, q \in e$ and $\Psi(p) = \Psi(q)$. By the definition of E_Γ , we can pick an edge \tilde{e} of G in $E_G \setminus E_G^{\text{col}}$ such that $\wp(\tilde{e}) = e$. Denote by \tilde{p} and \tilde{q} , preimages on \tilde{e} by \wp of p and q , respectively. We then have $\Phi(\tilde{p}) = \Phi(\tilde{q})$. According to the modified construction of Φ (Proposition 5.19), it follows that \tilde{p} and \tilde{q} belong to a path constituted by collapsing subdivision arcs of G . Hence $p = \wp(\tilde{p}) = \wp(\tilde{q}) = q$. It means that $\Psi|_e$ is injective. \square

5.10 The proof of $h(\Gamma, Q) = h(T, g)$

By Proposition 5.22, the map $\Psi : \Gamma \rightarrow T$ is surjective and finite to one. To prove $h(\Gamma, Q) = h(T, g)$, we only need to redefine Q on each subdivision arcs of Γ such that Ψ is a semi-conjugacy from $Q : \Gamma \rightarrow \Gamma$ to $g : T \rightarrow T$, and then apply Proposition 2.3.

Proposition 5.23. *The topological entropies $h(\Gamma, Q) = h(T, g)$.*

Proof. Remember that $\{\wp(\tilde{\delta}) \mid \tilde{\delta} \in \Delta_G \setminus \Delta_G^{\text{col}}\}$ forms a system of subdivision arcs for Γ . Let $\delta = \wp(\tilde{\delta})$ be such one. By the definition of the relation \simeq , the map $\wp : \tilde{\delta} \rightarrow \delta$ is a homeomorphism. By Proposition 5.19, and the commutative graphs (5.6) and (5.7), we have

$$g \circ \Psi(\delta) = g \circ \Phi(\tilde{\delta}) = \Phi \circ L(\tilde{\delta}) = \Psi \circ Q(\delta)$$

If $\Psi(\delta)$ is a singleton, the formula $g \circ \Psi = \Psi \circ Q$ naturally holds pointwise on δ . Otherwise, $Q(\delta)$ is an edge of Γ , and the maps $\Psi|_\delta$ and $\Psi|_{Q(\delta)}$ are both homeomorphisms onto their images. We then redefine Q on δ by lifting $g : \Psi(\delta) \rightarrow \Psi(Q(\delta))$ along these two homeomorphisms, i.e. by setting $Q := \Psi^{-1} \circ g \circ \Psi$ on δ . After such modification of Q on each subdivision arc of Γ , we obtain the formula $g \circ \Phi = \Phi \circ Q$ pointwise on Γ . By Proposition 2.5, the topological entropy $h(\Gamma, Q)$ is independent on the precise choices of Q as monotonous maps on the subdivision arcs of Γ . Then using Proposition 2.3, we have $h(\Gamma, Q) = h(T, g)$. \square

5.11 The proof of $h(G, L) = h(\Gamma, Q)$

To complete the proof of Theorem 1.2, it remains to show that $h(\Gamma, Q) = h(G, L)$. The idea of its proof is similar to that of Proposition 5.10.

Recall that $E_G^{\text{col}} = \{e \in E_G \mid \Phi(e) \text{ is a singleton}\}$. We have the following result.

Proposition 5.24. *For each $e \in E_G^{\text{col}}$, the image $L(e)$ is either one point or the union of edges in E_G^{col} .*

Proof. Let $e \in E_G^{\text{col}}$ and let $\delta \subset e$ be a subdivision arc of G . Then $\Phi(\delta)$ is a singleton. By Lemma 5.19, we have that $\Phi(L(\delta)) = g(\Phi(\delta))$ is a singleton. It follows that $L(\delta)$ is either a point or an edge of G belonging to E_G^{col} . \square

With this proposition, if we arrange the edges of G in the order $E_G^{\text{col}}, E_G \setminus E_G^{\text{col}}$, the incidence matrix of (G, L) takes the form

$$D_{(G,L)} = \begin{pmatrix} M & \star \\ \mathbf{0} & C \end{pmatrix}$$

where $\mathbf{0}$ is a zero matrix. Note that the matrix C is exactly the incidence matrix of (Γ, Q) , so it is enough to prove $\rho(M) = 1$.

From the modified definition of Φ , we know that an edge $e(x, y) \in E_G^{\text{col}}$ if and only if $\Phi(x) = \Phi(y)$, or equivalently, $\gamma_g(x) = \gamma_g(y)$. And by the definition ϕ , this happens if and only if $\gamma_f(x)$ and $\gamma_f(y)$ are contained in an essential \sim -class.

From now on and to the end, all argument take place in the f -plane, so we omit the subscript f in all quantities for simplicity.

Let K be a connected subset of K_f . An edge $e(x, y)$ of G is called **corresponding** to K if $\gamma(x), \gamma(y) \in K$. With this concept, any edge in E_G^{col} corresponds to an essential \sim -class.

Lemma 5.25. *Let $e = e(x, y) \in E_G^{\text{col}}$ correspond to a connected subset K of an essential \sim -class. Then all edges of G contained in $L(e)$ correspond to $f(K)$.*

Proof. Let the ordered pair x, y have the separation set (k_1, \dots, k_p) with $p \geq 0$, i.e., the leaf \overline{xy} successively crosses $\text{hull}(\Theta_{k_1}), \dots, \text{hull}(\Theta_{k_p})$ from x to y with $\Theta_{k_1}, \dots, \Theta_{k_p} \in \mathbf{M}$, and denote the subdivision arcs of G contained in $e(x, y)$ by $\delta_i, i = 0, \dots, p$, with $x \in \delta_0$ and $y \in \delta_{p+1}$.

Since K is allowable connected, then $[\gamma(x), \gamma(y)] \subset K$. By 2 of Lemma 5.13, we have $[\gamma(x), \gamma(y)] \cap H_{\Theta_{k_i}} \neq \emptyset$ for all $i \in \{1, \dots, p\}$. We claim that for each $i \in \{1, \dots, p\}$, the intersection of $[\gamma(x), \gamma(y)]$ and $H_{\Theta_{k_i}}$ contains a Julia point.

Recall that $r(\Theta_{k_i}) := \{r_{U,\theta} \mid U \text{ is a Fatou component with } \Theta(U) \subset \Theta_{k_i} \text{ and } \theta \in \Theta(U)\}$, and $H_{\Theta_{k_i}} = \cup_{r_{U,\theta} \in r(\Theta)} r_{U,\theta}$ (by 1 of Proposition 5.13). If $[\gamma(x), \gamma(y)]$ is disjoint from all critical Fatou components U with $\Theta(U) \in \Theta_{k_i}$, the claim is obviously true. Otherwise, let U be such a critical Fatou component intersecting $[\gamma(x), \gamma(y)]$. Since $\gamma(x), \gamma(y)$ are both Julia points, the intersection of U and $[\gamma(x), \gamma(y)]$ consists of two internal rays of U . They belong to $r(\Theta)$ because $[\gamma(x), \gamma(y)]$ is contained in a \sim -class. By 3 of Lemma 5.2, these two internal rays are $r_{U,\theta}$ and $r_{U,\eta}$ with $\theta, \eta \in \Theta(U)$. It follows that $\gamma(\theta), \gamma(\eta) \in [\gamma(x), \gamma(y)] \cap H_{\Theta_{k_i}}$.

From 2 of Lemma 5.7, we know that $H_{\Theta_{k_i}} \cap J_f = \{\gamma(\theta) \mid \theta \in \Theta_{k_i}\}$ for all $i \in \{1, \dots, p\}$. Then the claim above ensure that we can pick an argument θ_i in each Θ_{k_i} such that $\gamma(\theta_i) \in [\gamma(x), \gamma(y)] \subset K$. Setting $\theta_0 := x$ and $\theta_{p+1} := y$, we have $\gamma(\tau(\theta_0)), \dots, \gamma(\tau(\theta_{p+1})) \in f(K)$. By the definition of L , the edges of G contained in $L(e(x, y))$ are the non-trivial arcs among

$$L(\delta_0) = e(\tau(\theta_0), \tau(\theta_1)), \dots, L(\delta_p) = e(\tau(\theta_p), \tau(\theta_{p+1})).$$

They all correspond to $f(K)$. \square

The following lemma is quite similar to Lemma 5.9, both the statement and the effect.

Lemma 5.26. *All edges in E_G^{col} are attracted by cycles, i.e., for each $e \in E_G^{\text{col}}$, its iterations by L eventually fall on the union of cycles in E_G^{col} .*

Proof. Let e be an edge in E_G^{col} corresponding to an essential \sim -class ξ . By repeated use of Lemma 5.25, we get that for all $n \geq 1$, each edge of G contained in $L^n(e)$ corresponds to $f^n(\xi)$. By 4 of Lemma 5.7, there exists an $n_0 > 0$ such that for any $n \geq n_0$, the set $f^n(\xi)$ is either a periodic point in J_f or a periodic internal ray, or a star-like tree whose non-end vertex is a periodic point in J_f and every edge is a periodic internal ray. Hence $f^n(\xi) \cap J_f$ is a singleton, denoted by z_n . We just need to prove that for any $n \geq n_0$, each edge of G contained in $L^n(e)$ is periodic by the iterations of L .

Let $n \geq n_0$. Denote by A_n the set of external angles associated with z_n . We claim that each pair of angles in A_n are non-separated by Θ . Suppose on the contrary that $x, y \in A_n$ are separated by an element of \mathbf{M} . By Lemma 4.7, they are also separated by a portrait leaf $\overline{\theta\eta}$ of Θ . The corresponding portrait cutting line $\mathcal{R}(\theta, \eta)$ thus contains z_n , and separates $\mathcal{R}(x), \mathcal{R}(y)$. Since z_n is periodic, the angles θ, η belong to a Fatou-type element $\Theta(U)$ of Θ , and one of them, say θ , belongs to A_n . It contradicts that $\mathcal{R}(\theta)$ supports the Fatou component U at z_n .

Let $e(x, y)$ be an edge of G contained in $L^n(e)$. We know from the discussion above that $e(x, y)$ corresponds to $f^n(\xi)$. It means that $\gamma(x)$ and $\gamma(y)$ belong to $f^n(\xi) \cap J_f = \{z_n\}$. By the claim above, we get $L(e(x, y)) = e(\tau(x), \tau(y))$, which is also an edge of G and corresponds to $f^{n+1}(\xi)$. Since this argument holds for all sufficiently large n , and the angles in A_n are periodic (because z_n is periodic), the edge $e(x, y)$ must also be periodic by iterations of L . \square

Proposition 5.27. *The topological entropies verify $h(\Gamma, Q) = h(G, L)$.*

Proof. The effect of Lemma 5.26 in this proof is the same as that of Lemma 5.9 in the proof of Proposition 5.10. Using a similar argument, just replacing H_f and f in the proof of Proposition 5.10 with G and L , respectively, we obtain the equation $h(G, L) = h(\Gamma, Q)$. We omit the details. \square

Proof of Theorem 1.2. Combining Propositions 4.11, 5.10, 5.23 and 5.27, we get that $h(H_f, f) = \log \rho(\Theta)$. \square

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